FOURIER INTEGRAL TRANSFORMS

Fourier integral transforms are generalizations of Fourier series.

For a function $f(x)$ defined on an interval $[-L, L]$, we have the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(y)\cos\left(\frac{n\pi y}{L}\right) dy \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(y)\sin\left(\frac{n\pi y}{L}\right) dy \ .$$

An alternative way of writing these series is to use the formulation of the trigonometric functions in terms of the complex exponential function.

$$\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \end{aligned}$$

This gives

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \frac{1}{2}\left(a_n - ib_n\right)e^{in\pi x/L} + \frac{1}{2}\left(a_n + ib_n\right)e^{-in\pi x/L} \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \end{aligned}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) \\ &= \frac{1}{2L}\int_{-L}^{L} f(y)\left(\cos\left(\frac{n\pi y}{L}\right) - i\sin\left(\frac{n\pi y}{L}\right)\right) dy \\ &= \frac{1}{2L}\int_{-L}^{L} f(y)e^{-in\pi y/L}\, dy \end{aligned}$$

Combining these we obtain

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{\infty}\int_{-L}^{L} f(y)\exp\left(i\frac{n\pi(x-y)}{L}\right) dy$$

Setting $h = \pi/L$, we obtain

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\left\{ h\sum_{-\infty}^{\infty}\int_{-L}^{L} f(y)\exp(i(x-y)(nh))\, dy \right\} \\ &= \frac{1}{2\pi}\sum_{-\infty}^{\infty} hF(nh) \end{aligned}$$

where

$$F(\omega) = \int_{-L}^{L} f(y)\exp(i(x-y)\omega)\,dy\ .$$

Now,

$$\sum_{-N}^{N} hF(nh) \sim \int_{-Nh}^{Nh} F(\omega)\,d\omega\ ,$$

so that

$$\sum_{-\infty}^{\infty} hF(nh) \sim \int_{-\infty}^{\infty} F(\omega)\,d\omega\ ,$$

an approximation which improves as $h \to 0$.

Letting $L \to \infty$ (and therefore $h \to 0$), we obtain

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left(\int_{-\infty}^{\infty} f(y)\exp i(x-y)\omega\,dy\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega x}\left(\int_{-\infty}^{\infty} dy\, e^{-i\omega y} f(y)\right) \end{aligned}$$

The pair of equations

$$\begin{aligned} \mathcal{F}(\omega) = &\int_{-\infty}^{\infty} f(y)e^{-i\omega y}\,dy \ = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\ f(x) = &\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)e^{i\omega x}\,d\omega \end{aligned}$$

are known as the Fourier Transform pair.

The first equation defines the Fourier Transform of the function f, while the second defines the inverse transform. This is the form in which the transform usually appears, and which will be used in this course. However, you should be aware that some authors prefer the symmetric definition

$$\begin{aligned} \hat{\mathcal{F}}(\omega) = &\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\ f(x) = &\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\mathcal{F}}(\omega)e^{i\omega x}\,d\omega \end{aligned}$$

(This form can be recognised in practice by the extraneous factors of $\sqrt{2\pi}$.)

e.g. $f(x) = e^{-|x|}$.

$$\begin{aligned} \mathcal{F}(\omega) &= \int_{-\infty}^{\infty} e^{-|y|-i\omega y}\,dy \\ &= \int_{-\infty}^{0} e^{(1-i\omega)y}\,dy + \int_{0}^{\infty} e^{-(1+i\omega)y}\,dy \\ &= \frac{1}{1-i\omega}e^{(1-i\omega)y}\Bigg|_{-\infty}^{0} - \frac{1}{1+i\omega}e^{-(1+i\omega)y}\Bigg|_{0}^{\infty} \\ &= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2} \end{aligned}$$

$$f(x) = \frac{1}{x^2+1}$$

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega y}}{y^2+1}\,dy$$

Since $f(z) = 1/(z^2+1)$ goes to 0 as $|z| \to \infty$, we can use Jordan's Lemma to evaluate this integral.

When $\omega < 0$, we can close the contour in the upper half complex plane, so that

$$\begin{aligned}\mathcal{F}(\omega) &= 2\pi i \times \text{ Residue at} i\\ &= 2\pi i \frac{e^{-i\omega i}}{2i}\\ &= \pi e^{\omega}\end{aligned}$$

When $\omega > 0$, we can close the contour in the lower half complex plane, so that

$$\begin{aligned}\mathcal{F}(\omega) &= -2\pi i \times \text{ Residue at } -i\\ &= -2\pi i \frac{e^{-i\omega(-i)}}{-2i}\\ &= \pi e^{-\omega}\end{aligned}$$

Note that this example also verifies the inversion formula for the first example.

Laplace Transforms

In order for the Fourier Transform Integral to converge, we require that $|f| \to 0$ as $|x| \to \infty$.

Suppose that we are given a function $g(x)$ which is identically 0 for $x < 0$, and for which we can find a real constant c such that

$$\int_0^{\infty} e^{-cx} g(x)\,dx$$

exists.

Then the function $f(x) = e^{-cx} g(x)$ will have a Fourier transform.

$$\begin{aligned}\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx\\ &= \int_0^{\infty} e^{-(c+i\omega)x} g(x)\,dx\\ &= \int_0^{\infty} e^{-px} g(x)\,dx\end{aligned}$$

where $p = c + i\omega$. The function $G(p) = \mathcal{F}(\omega)$ is called the Laplace transform of g.

From the inversion formula, we have

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\mathcal{F}(\omega)\,d\omega \\
e^{-cx}g(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\mathcal{F}(\omega)\,d\omega \\
g(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(c+i\omega)x}\mathcal{F}(\omega)\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{px}G(p)\,d\omega \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}G(p)\,dp
\end{aligned}$$

This integral can frequently be evaluated for $x > 0$ by closing the contour in the left half plane.

Note that $G(p)$ has no poles for $\mathrm{Re}(p) > c$, so that for $x < 0$ closing the contour in the right half plane gives $g(x) = 0$.