Special/Supplementary Examination, August 1998

(Complex Variables)

Time: **TWO** (2) **hours** for working

1. Solve the equation

$$z^5 = 1$$

and plot the roots in the complex plane.

The function $f(t)$ is periodic with period 1, and

$$f(0) = 0.4 \ ; \ f(0.2) = 0.6 \ ; \ f(0.4) = 0.8 \ ; \ f(0.6) = 0.0 \ ; \ f(0.8) = 0.2 \ .$$

Use the Discrete Fourier Transform to derive an approximation of the form

$$f(t) \sim c_0 + c_1 \cos(2\pi t - \phi_1) + c_2 \cos(4\pi t - \phi_2) \ .$$

What value does this approximation give for $f(0.5)$?

2. The function $f(z) = u(x,y) + iv(x,y)$ from $\mathbb{C}$ to $\mathbb{C}$ is differentiable at the point $z_0 = x_0 + iy_0$.

Show that

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \ .$$

Determine the points at which the function

$$f(z) = \bar{z}^2$$

is differentiable.

3. Define the function $\exp(z)$, where $z = x + iy$, in terms of real functions of x and y.

Hence express $\cos z$ in the form $u(x,y) + iv(x,y)$.

Find all roots of the equation

$$\cos z = 2 \ .$$

4. (a) The contour C is the straight line from $z = 0$ to $z = 2 + i$.

Evaluate

$$\int_C \bar{z} \, dz \ .$$

(b) By considering

$$\oint_C e^{-z^2} \, dz$$

around the rectangular contour C with vertices $\pm R$, $\pm R - \frac{1}{2} i\alpha$, evaluate

$$\int_{-\infty}^{\infty} e^{i\alpha x} e^{-x^2} \, dx \ .$$

(You may take as given that $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$.)

5.(a) The function $f(z)$ is regular inside and on the circle $C : |z - z_0| = R$. Prove that, for $|z - z_0| < R$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \ .$$

(b) Determine the Laurent expansion of the function

$$f(z) = \frac{z}{(z-1)(z-2)}$$

in the annulus $1 < |z| < 2$.

6. Evaluate

(a)
$$\int_{-\pi}^{\pi} \frac{d\theta}{3 + 2\cos\theta}$$

(b)
$$\int_{0}^{\infty} \frac{dx}{(1 + x^3)}$$

7. The function $f(z)$ is regular for $|z - a| \leq R$, and $|f(z)| \leq M$ on the circle $|z - a| = R$.

Show that the equation
$$z = a + \zeta\, f(z)$$
has one root in the circle $|z - a| < R$ for $|\zeta| < R/M$.

Show that this root can be expressed as

$$z = a + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \, D^{n-1} \left(f(z)\right)^n \Big|_{z=a} \ .$$

8. Derive the z-transform of the sequence $\{a_n = r^n\}$.

Use the z-transform to solve the difference equation

$$x_{n+2} - 2x_{n+1} - 3x_n = 1 \ ; \ n \geq 0$$
$$x_0 = 1 \ , \ x_1 = 2$$