First Semester Examination, June 2000

Complex Analysis III

Time: TWO (2) **hours** for working

1. The function $f(z) = u(x,y) + iv(x,y)$ from $\mathbb{C}$ to $\mathbb{C}$ is differentiable at the point $z_0 = x_0 + iy_0$.

Show that

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0) \ .$$

Define the function $\exp(z)$, where $z = x + iy$, in terms of real functions of x and y.

Hence express $\sin z$ in the form $u(x,y) + iv(x,y)$.

Find all roots of the equation

$$\sin z = 2 \ .$$

2. (a) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$$

(b) Find the Fourier transform of

$$\frac{x}{1+x^2} \ .$$

3. Let $\mathcal{D}$ be the annular domain between the two circles

$$C_1 = \{|z| = 1\} \quad \text{and} \quad C_2 = \left\{|z-1| = \frac{5}{2}\right\} \ .$$

(a) Find a linear fractional transformation

$$w = \frac{az+b}{cz+d}$$

which maps C_1 and C_2 onto concentric circles in the w-plane.

(b) Hence find the function $\phi(x,y)$ such that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{in } \mathcal{D} \ ;$$
$$\phi = 0 \text{ on } C_1 \ ; \ \phi = 1 \text{ on } C_2$$

4. (a) The function $f(z)$ is regular inside and on the simple, closed, rectifiable oriented curve (scroc) C.

If z_0 is in the interior of C, show that

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \ .$$

(b) Use residues to evaluate

(i) $$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx \ ;$$

(ii) $$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} \ .$$

5. Let C_N be the square contour with vertices $(\pm(N+\frac{1}{2}) \pm i(N+\frac{1}{2}))$, where N is a positive integer.

Show that the function $\pi\cot(\pi z)$ is bounded on C_N for every N.

Evaluate

$$\oint_{C_N} \frac{\pi\cot(\pi z)}{z^2+1}\, dz \ ,$$

and hence determine the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \ .$$