The exponential function

We have seen the representation

$$\cos\theta + i\sin\theta = e^{i\theta} \ .$$

We use this to define the exponential function for an arbitrary complex number:

$$\begin{aligned}\exp(z) = e^z &= e^{x+iy} = e^x e^{iy} \\ &= e^x \cos y + i e^x \sin y\end{aligned}$$

(Remember when evaluating this function that y has radian measure.)

From the definition we have immediately the following properties;

(i) $e^0 = 1$.

(ii) $|e^z| = e^x$.

(iii) e^z is not zero for any z.

(iv) $1/e^z = e^{-z}$.

(v) $(e^z)^n = e^{nz}$.

(vi) $e^{\pi i} = -1$ and $e^{2\pi i} = 1$.
Hence, $e^{z+2\pi i} = e^z$;
the exponential function is periodic
with period $2\pi i$.

Graphical representation of the exponential function

If we consider the domain $\mathcal{D}$ consisting of the infinite strip

$$-\pi < \mathrm{Im} z < \pi$$

then the exponential function maps this domain one-to-one onto the complex plane with the exception of the negative real axis.

The strip is called a *fundamental strip* for the exponential function, and the region onto which it is mapped is called a *cut plane.*

Points just above the cut correspond to

$$\exp(a + i(\pi -))$$

while points below the cut correspond to

$$\exp(a + i(-\pi +))$$

The cut is also called a *branch cut*, and the ends of the cut ($w = 0$ and $w = \infty$) are called *branch points.*

Choosing a different fundamental strip,

$$b < \mathrm{Im} z < b + 2\pi$$

produces a different branch cut in general, but does not change the branch points.

If we choose the strip

$$\pi < \text{Im}z < 3\pi$$

then the cut again lies along the negative real axis. In this case, points just below the cut correspond to

$$\exp(a + i(\pi+))$$

and can be considered as the continuation of of the points above the cut in the first case.

If we join these two cut planes appropriately across the cut, we have a *spiral staircase* like structure which represents the map of the double strip

$$-\pi < \text{Im}z < 3\pi$$

and we could continue in like fashion to add cut planes top and bottom to create an infinite spiral which is the map of $\mathbb{C}$ under the exponential function.

This structure is an example of a *Riemann surface*.