The following result is due to Euler. It provides a nice application of elementary complex number theory.

We wish to prove:

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

We use the representation of the sin function in terms of the complex exponential. In order to simplify the notation, we also set $\alpha = e^{i\pi/n}$ and $\omega = \alpha^{-2}$. Therefore,

$$\sin\left(\frac{k\pi}{n}\right) = \frac{1}{2i}\left(e^{ik\pi/n} - e^{-ik\pi/n}\right) = \frac{1}{2i}\left(\alpha^k - \alpha^{-k}\right)$$

and

$$\begin{aligned}\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) &= \frac{1}{2^{n-1}i^{n-1}}\prod_{k=1}^{n-1}(\alpha^k - \alpha^{-k}) \\ &= \frac{\alpha^{(1+2+\cdots+n-1)}}{2^{n-1}i^{n-1}}\prod_{k=1}^{n-1}(1-\alpha^{-2k}) \\ &= \frac{1}{2^{n-1}}\prod_{k=1}^{n-1}(1-\omega^k)\end{aligned}$$

since

$$\alpha^{(1+2+\cdots+n-1)} = \alpha^{(n-1)n/2} = e^{i(n-1)\pi/2} = i^{n-1} \ .$$

Now, $\{\omega, \omega^2, \ldots, \omega^{n-1}\}$ are the non-trivial n^{th} roots of unity. Therefore they satisfy the equation $p(z) = (z^n - 1)/(z-1) = 0$.

Therefore

$$\prod_{k=1}^{n-1}(z - \omega^k) = p(z) = z^{n-1} + z^{n-2} + \cdots + 1 \ .$$

Setting $z = 1$ gives

$$\prod_{k=1}^{n-1}(1-\omega^k) = n$$

as required.