Elementary Conformal Mappings

1. Translations:

$$w = z + c$$

Every point in $\mathbb{C}$ is moved the same distance in the same direction.

Figures are preserved in both shape and size.

The mapping is *congruent*.

The inverse mapping is the translation

$$z = w - c$$

2. Rotations:

$$w = e^{i\theta} z$$

The complex plane is rotated uniformly about the origin.

Figures are preserved in both shape and size.

The mapping is *congruent* .

The inverse mapping is the rotation

$$z = e^{-i\theta} w$$

3. Dilations:

$$w = rz$$

where r is real and positive.

Every part of $\mathbb{C}$ is magnified by a factor r.

The mapping preserves shapes but not size (unless $r = 1$.)

This is a *similarity* transformation.

The inverse mapping is the dilation

$$z = \frac{1}{r} w$$

4. The general linear transformation

$$w = az + b \ , \quad a \neq 0$$

Writing this as

$$\begin{aligned} w &= az + b \\ &= a(z + b/a) \\ &= |a| e^{i\theta} (z + b/a) \end{aligned}$$

we see that this transformation is, in general, a combination of the three mapping already considered.

$$\begin{aligned} \zeta_1 &= z + b/a \\ \zeta_2 &= e^{i\theta} \zeta_1 \\ w &= |a| \zeta_2 \end{aligned}$$

Therefore, this mapping preserves shapes, but does not preserve sizes unless $|a| = 1$.

The inverse mapping is the linear transformation

$$z = \frac{1}{a}w - \frac{b}{a}$$

Each of these mapping is a $1-1$ mapping of $\mathbb{C}$ onto itself.

Conversely, suppose that $w = f(z)$ is a regular $1-1$ mapping of $\mathbb{C}$ onto itself.

Since f is defined and regular for all $\mathbb{C}$, f is entire, and its Taylor series expansion converges for all z.

Suppose

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

has infinitely many non-zero terms in the expansion.

Then

$$F(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^{-k}$$

converges for all $|\zeta| > 0$, and represents a function with an essentail singularity at $\zeta = 0$.

This means that $F(\zeta)$ takes every value in $\mathbb{C}$ (except possibly 2) infinitely often, and therefore so does $f(z)$.

($f(z)$ has an *essential singularity at infinity*.)

Since we have assumed that f is a $1-1$ mapping, this case cannot happen.

Therefore the Taylor series expansion has only a finite number of non-zero terms, and $f(z)$ is a polynomial in z.

If the degree of f is n, then $f'(z)$ is a polynomial of degree $n-1$ in z.

By the fundamental theorem of algebra, f' will have $n-1$ zeros in $\mathbb{C}$.

But if f is $1-1$, $f' \neq 0$ in $\mathbb{C}$.

Therefore $n = 1$, and

$$f(z) = az + b$$

where $a \neq 0$.

Therefore the most general regular $1-1$ mapping of $\mathbb{C}$ onto itself is the linear transformation.

5. Inversion:

$$w = \frac{1}{z}$$

This mapping is $1-1$ except for the values $z = 0$ and $w = 0$.

We define the *point at infinity* as the image of the origin under this mapping;

$$\infty = \frac{1}{0}$$
$$0 = \frac{1}{\infty}$$

If we add this point to the complex plane, we get the *extended complex plane*; $\mathbb{C} \cup \{\infty\}$.

This is most conveniently represented by the *Riemann sphere*.

If you imagine a unit sphere centered at the origin in the complex plane, then the straight line joining the North pole of the sphere to a point on the complex plane intersects the sphere at a unique point.

This *stereographic projection* defines a $1-1$ mapping of the complex plane $\mathbb{C}$ onto the points of the sphere excluding the north pole.

If we now identify the north pole with the point at infinity, we have a $1-1$ mapping of the extended complex plane onto the sphere.

Technically, this is called a *one-point compactification*.

The mapping

$$w = \frac{1}{z}$$

is a $1-1$ mapping of the sphere onto itself.

It preserves the shapes and sizes of figures on the sphere, but not of figures in the plane in general.

However, consider the circle

$$\begin{gathered} |z - z_0| = r \\ |z - z_0|^2 = (z - z_0)(z^* - z_0^*) = r^2 \\ zz^* - z_0 z^* - z_0^* z + z_0 z_0^* - r^2 = 0 \end{gathered}$$

Under inversion it maps onto

$$\begin{gathered} \frac{1}{w}\frac{1}{w^*} - \frac{z_0}{w^*} - \frac{z_0^*}{w} + (|z_0|^2 - r^2) = 0 \\ (|z_0|^2 - r^2)ww^* - z_0 w - z_0^* w^* + 1 = 0 \end{gathered}$$

Provided $|z_0|^2 \neq r^2$, which corresponds to the original circle passing through the origin in the z-plane, we can divide through to get

$$\begin{gathered} ww^* - az_0 w - az_0^* w^* + a = 0 \\ (\ a = 1/(|z_0|^2 - r^2)\ ) \\ (w - az_0^*)(w^* - az_0) = a^2|z_0|^2 - a \\ = a^2(|z_0|^2 - 1/a) = r^2 a^2 = R^2 \\ |w - az_0^*| = R = r|a| \end{gathered}$$

Therefore, in general the inversion maps circles onto circles.

In the exceptional case when the original circle passes through the origin, the image is

$$\begin{gathered} z_0 w + z_0^* w^* = 1 \\ 2x_o u - 2y_0 v = 1 \end{gathered}$$

that is, a straight line.

Since the inverse transform of $w = 1/z$ is $z = 1/w$, it follows that the image of a straight line

$$ax + by = c$$

will be a circle unless $c = 0$.

This exception corresponds to a straight line passing through the origin, whose image is another straight line through the origin;

$$ax + by = 0 \quad \rightarrow \quad au - bv = 0$$

Straight lines are considered as *'circles of infinite radius'*.

In summary: the mapping

$$w = \frac{1}{z}$$

maps (circles/straight lines) in the z-plane onto (circles/straight lines) in the w-plane.

The image is a circle unless the original curve passes through the origin in the z-plane.

The Linear Fractional Transformation

Also known as the *bilinear transformation* or the *Möbius Transformation*.

Consider a combination of linear mappings with an inversion.

$$\zeta_1 = cz + d$$
$$\zeta_2 = \frac{1}{\zeta_1}$$
$$w = \alpha\zeta_2 + \beta$$
$$w = \frac{\alpha}{cz+d} + \beta$$
$$= \frac{\alpha + c\beta z + d\beta}{cz+d} = \frac{az+b}{cz+d}$$
$$\text{Note: } ad - bc = c\beta d - \alpha c - d\beta c = -\alpha c \neq 0$$

Since the inversion sends (circles/straight lines) into (circles/straight lines) and the linear transformations preserve shapes, this mapping also sends (circles/straight lines) into (circles/straight lines).

The image is a circle unless the original passes through the origin in the ζ_1-plane; i.e. the point $z = -d/c$ in the z-plane.

The transformation is unchanged if we multiply all the coefficients by a non-zero constant

$$\frac{az+b}{cz+d} = \frac{raz+rb}{rcz+rd}$$

Therefore for *theoretical* purposes it is convenient to assume that the coefficients have been normalised so that $ad - bc = 1$.

However, this restriction is usually ignored in practical examples.

The combination of two LF transformations;

$$\zeta = \frac{a_1 z + b_1}{c_1 z + d_1}$$
$$w = \frac{a_2\zeta + b_2}{c_2\zeta + d_2}$$
$$= \frac{a_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + b_2}{c_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + d_2}$$
$$= \frac{a_2(a_1 z + b_1) + b_2(c_1 z + d_1)}{c_2(a_1 z + b_1) + d_2(c_1 z + d_1)}$$
$$= \frac{(a_2 a_1 + b_2 c_1)z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1)z + (c_2 b_1 + d_2 d_1)}$$

This is another linear fractional transformation.

The coefficients satisfy

$$\begin{pmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{pmatrix}$$
$$= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

In particular, the inverse of

$$w = \frac{az + b}{cz + d}$$

is

$$z = \frac{dw - b}{-cw + a}$$

Therefore the set of all linear fractional transformations form a group, isomorphic to the set of all 2×2 complex unimodal (i.e. the determinant $= 1$) matrices.

This group contains all $1 - 1$ maps of the extended complex plane onto itself.