Conformal mapping

Consider a smooth curve $\mathcal{C}$;

$$(X(t), Y(t)) \quad \text{in } \mathbb{R}^2$$

or equivalently

$$Z(t) = X(t) + iY(t) \quad \text{in } \mathbb{C}$$

where t is a real parameter.

e.g. The parabolic segment

$$\mathcal{C}: \quad Z(t) = t + it^2 \quad -1 \leq t \leq 1$$

If $z_0 = Z(t_0)$ and $z_1 = Z(t_1)$, $(t_0 < t_1)$,
are two points on the curve, then the angle subtended by the line segment

$$Z(t_1) - Z(t_0)$$

with the real axis is

$$\arg(Z(t_1) - Z(t_0))$$

Since the argument is unaffected by multiplication by a real constant, this angle is also given by

$$\arg\left(\frac{Z(t_1) - Z(t_0)}{t_1 - t_0}\right)$$

Taking the limit as t_1 approaches t_0, we see that the angle between the positive tangent at z_0 and the real axis is

$$\arg(\dot{Z}(t_0))$$

e.g. $Z(t) = t + it^2$.

$$\dot{Z}(t) = 1 + 2it$$
$$\dot{Z}(0.5) = 1 + i = \sqrt{2}e^{i\pi/4}$$

The tangent at $(1, \frac{1}{4})$ subtends and angle of $\frac{\pi}{4}^c$ or 45^o with the real axis.

Now consider the curve $\mathcal{C}^* = f(\mathcal{C})$ generated by the regular mapping

$$w = f(z) = u(x, y) + iv(x, y) \ .$$

It has the parametric representations

$$(U(t), V(t)) \quad \text{in } \mathbb{R}^2$$

or equivalently

$$W(t) = U(t) + iV(t) \quad \text{in } \mathbb{C}$$

where

$$U(t) = u(X(t), Y(t)) \ ; \ V(t) = v(X(t), Y(t))$$

e.g. $w = e^z$ maps the parabolic arc $Z = t + it^2$ onto

$$W(t) = e^t \cos(t^2) + ie^t \sin(t^2)$$

e.g. $w = z^2$ maps the parabolic arc $Z = t + it^2$ onto

$$W(t) = (t^2 - t^4) + i2t^3$$

At $w_0 = f(z_0)$, the angle between the positive tangent to the new curve and the real axis is

$$\begin{aligned} \arg(\dot{W}(t_0)) &= \arg\Big(f'(z_0)\dot{Z}(t_0)\Big) \\ &= \arg(f'(z_0)) + \arg(\dot{Z}(t_0)) \end{aligned}$$

provided $f'(z_0) \neq 0$.

Suppose that $Z_1(t)$ and $Z_2(s)$ are two curves passing through z_0.
The angle of the intersection between these curves is given by

$$\arg(\dot{Z}_2(s_0)) - \arg(\dot{Z}_1(t_0))$$

If we map these curves onto $W_1(t)$ and $W_2(s)$ by the regular mapping $w = f(z)$, where $f'(z_0) \neq 0$, then the angle of the intersection between W_1 and W_2 at $w_0 = f(z_0)$ is

$$\begin{aligned} &\arg(\dot{W}_2(s_0)) - \arg(\dot{W}_1(t_0)) \\ = &\Big(\arg(f'(z_0)) + \arg(\dot{Z}_2(s_0))\Big) - \Big(\arg(f'(z_0)) + \arg(\dot{Z}_1(t_0))\Big) \\ &= \arg(\dot{Z}_2(s_0)) - \arg(\dot{Z}_1(t_0)) \end{aligned}$$

Provided $f'(z) \neq 0$, the mapping preserves angles in magnitude and orientation.
We say that the mapping is *conformal*.

The local behaviour of a smooth mapping

$$(x, y) \to (u(x, y), v(x, y))$$

from $\mathbb{R}^2$ to $\mathbb{R}^2$ is given by

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

This mapping is locally invertible provided the Jacobian

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0$$

If the mapping is generated by a regular function $w = f(z)$, then the Cauchy-Riemann equations are satisfied, and

$$\begin{aligned} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} &= u_x v_y - v_x u_y \\ &= (u_x)^2 + (v_x)^2 \\ &= |f'(z)|^2 \end{aligned}$$

Therefore the condition for invertibility is the same as that for conformality; namely $f'(z) \neq 0$.

The inverse function theorem for regular functions.

Suppose that $f(z)$ is regular and $f'(z_0) \neq 0$.

Setting

$$\begin{aligned} w &= f(z) \\ w_0 &= f(z_0) \end{aligned}$$

we want to express z as a function of w for w near w_0.

Specifically, we will show that we can write

$$z = z_0 + \sum_{n=1}^{\infty} c_n (w - w_0)^n$$

We have

$$\begin{aligned} w - w_0 &= f(z) - f(z_0) \\ &= \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) \\ &= \frac{z - z_0}{\phi(z)} \end{aligned}$$

where

$$\phi(z) = \begin{cases} \frac{z - z_0}{f(z) - f(z_0)} & z \neq z_0 \\ \frac{1}{f'(z_0)} & z = z_0 \end{cases}$$

Rearranging the equation we have

$$\begin{aligned} z - z_0 &= (w - w_0)\phi(z) \\ z &= z_0 + (w - w_0)\phi(z) \end{aligned}$$

which is the form required for a Lagrange expansion.

Therefore

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(w - w_0)^n}{n!} \left. \frac{d^{n-1}}{dt^{n-1}} (\phi(t))^n \right|_{t=z_0}$$

e.g.

$$w = ze^z$$

Near $z_0 = 0$, $w_0 = 0$, we have

$$\begin{aligned} z &= we^{-z} \\ &= \sum_{n=1}^{\infty} \frac{w^n}{n!} \left(D^{n-1}(e^{-nz})\big|_{z=0} \right) \\ &= \sum_{n=1}^{\infty} (-n)^{n-1} \frac{w^n}{n!} \end{aligned}$$

The radius of convergence of this series is

$$\begin{aligned}
\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|} &= \lim_{n\to\infty}\frac{n^{n-1}}{n!}\frac{(n+1)!}{(n+1)^n} \\
&= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n-1} \\
&= \frac{1}{e}
\end{aligned}$$

which represents the distance from $w = 0$ to the point at which $w'(z) = 0$.

Returning to the local representation of the mapping;
when $f'(z) \neq 0$, we can define θ by

$$\begin{aligned}
u_x &= |f'(z)| \cos\theta \\
v_x &= |f'(z)| \sin\theta
\end{aligned}$$

so that the matrix of the local linear transformation becomes

$$\begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = |f'(z)| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

showing that locally the mapping is represented by rotation through the angle θ and magnification by the factor $|f'(z)|$.

Conversely:
Suppose that the mapping

$$(x, y) \to (u(x, y), v(x, y))$$

is conformal.
In the neighbourhood of a typical point (x_0, y_0), the transformation is represented by the Jacobian matrix

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Note that since a conformal mapping is necessarily locally $1-1$, the matrix J is non-singular.

Therefore there are a positive definite symmetric matrix A and an orthogonal matrix U such that

$$J = A\,U$$

Since A is symmetric,

(a) the eigenvalues of A are real;

(b) the corresponding eigenvectors are orthogonal

positive definite means that the eigenvalues λ_1 and λ_2 are positive.
Since

$$|A| = \lambda_1\,\lambda_2$$

the determinant of A is also positive.

Since U is orthogonal,

it represents a rotation if $|U| = 1$;

it represents a reflection if $|U| = -1$.

If the mapping is conformal, $|J| > 0$.

However, $|J| = |A|\,|U|$, and since $|A| > 0$, $|U| = 1$.

Therefore

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some angle θ.

Now consider the matrix A.

Since the mapping is conformal, it preserves angles.

The rotation U obviously preserves angles, so that for conformality A must also preserve angles.

Take the orthogonal eigenvectors e_1 and e_2 of A, normalised to have unit length.

The angle between e_1 and $e_1 + e_2$ is 45^o.

These vectors are mapped onto $\lambda_1 e_1$ and $\lambda_1 e_1 + \lambda_2 e_2$ by A.

The tangent of the angle between these vectors is λ_2/λ_1.

Since the mapping is conformal, the angle is 45^o, and the value of the tangent is 1,

Therefore

$$\lambda_1 = \lambda_2$$

and

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

This gives

$$J = \begin{pmatrix} \lambda_1\cos\theta & -\lambda_1\sin\theta \\ \lambda_1\sin\theta & \lambda_1\cos\theta \end{pmatrix}$$

so that we have

$$u_x = v_y$$
$$v_x = -u_y$$

The functions $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemmann equtaions, so that the function

$$f(z) = u(x,y) + iv(x,y)$$

is regular.

The behaviour when $f'(z) = 0$ is illustrated by considering the functions $w = z^n$.

If we consider the sector bounded by the lines $\arg(z) = 0$, $|z| = r$ and $\arg(z) = \alpha$, it is mapped onto the sector bounded by $\arg(w) = 0$, $|w| = r^n$ and $\arg(w) = n\alpha$.

Therefore the angle at the origin, where $f' = 0$, is magnified by a factor n, while the local neighbourhood shrinks towards the origin.

We can keep a $1 - 1$ mapping provided any point at which $f'(z) = 0$ is a corner point of the region being mapped, and the angle at the corner is less than $2\pi/n$.

In particular, the mapping $w = z^2$ maps the quadrant $x \geq 0$, $y \geq 0$ $1 - 1$ onto the upper half plane $v \geq 0$, and the mapping is conformal except at the origin.