$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{p}}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{px}}{\sqrt{p}}\,dp$$

Since the integrand has a branch point at the origin, we cannot close the contour directly by using the Bromwich contour.

Instead, we use a combination of the Bromwich contour and the keyhole contour.

Taking $\arg(p) = 0$ when p is real and positive, we define the contour C as follows:

(a) $$p = c + iy \;;\; -\sqrt{R^2 - c^2} \le y \le \sqrt{R^2 - c^2}$$

(b) $$p = Re^{i\theta} \;;\; \frac{\pi}{2} - \arcsin\left(\frac{c}{R}\right) \le \theta \le \pi$$

(c) $$p = se^{i\pi} \;;\; R \ge s \ge \epsilon$$

(d) $$p = \epsilon e^{i\theta} \;;\; \pi \ge \theta \ge -\pi$$

(e) $$p = se^{-i\pi} \;;\; \epsilon \le s \le R$$

(f) $$p = Re^{i\theta} \;;\; -\pi \le \theta \le -\frac{\pi}{2} + \arcsin\left(\frac{c}{R}\right)$$

We divide the integral along (b) into two sections;

$$\frac{\pi}{2} - \arcsin\left(\frac{c}{R}\right) \le \theta \le \frac{\pi}{2}$$
$$\text{and} \qquad \frac{\pi}{2} \le \theta \le \pi$$

On the first piece, $\cos\theta \le \frac{\pi}{2} - \theta$, and we have

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int \frac{e^{px}}{\sqrt{p}}\,dp\right| &\le \frac{1}{2\pi}\int \frac{e^{R\cos\theta x}}{\sqrt{R}} R\,d\theta \\
&\le \frac{1}{2\pi}\sqrt{R}\int e^{Rx(\frac{\pi}{2}-\theta)}\,d\theta \\
&= \frac{1}{2\pi}\sqrt{R}\left(-\frac{1}{Rx}e^{Rx(\frac{\pi}{2}-\theta)}\right)\Bigg| \\
&= \frac{1}{2\pi}\frac{1}{x\sqrt{R}}\left(e^{Rx\arcsin(c/R)} - 1\right)
\end{aligned}$$

As $R \to \infty$, $R\arcsin(c/R) \to c$, so that this contribution tends to 0 as $R \to \infty$.

On the second piece, $\cos\theta \le 1 - \frac{2}{\pi}\theta$, and we have

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int \frac{e^{px}}{\sqrt{p}}\,dp\right| &\le \frac{1}{2\pi}\int \frac{e^{R\cos\theta x}}{\sqrt{R}} R\,d\theta \\
&\le \frac{1}{2\pi}\sqrt{R}\int e^{Rx(1-2\theta/\pi)}\,d\theta \\
&= \frac{1}{2\pi}\sqrt{R}\left(-\frac{\pi}{2Rx}e^{Rx(1-2\theta/\pi)}\right)\Big| \\
&= \frac{1}{4x\sqrt{R}}\left(1-e^{-Rx}\right)
\end{aligned}$$

which also tends to 0 as $R \to \infty$.

Similar calculations apply to section (f).

On (d), it is sufficient to use $\cos\theta \le 1$.

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int \frac{e^{px}}{\sqrt{p}}\,dp\right| &\le \frac{1}{2\pi}\int \frac{e^{\epsilon\cos\theta x}}{\sqrt{\epsilon}}\epsilon\,d\theta \\
&\le \frac{1}{2\pi}\sqrt{\epsilon}\,2\pi\,e^{\epsilon x} \\
&= \sqrt{\epsilon}\,e^{\epsilon x}
\end{aligned}$$

which tends to 0 as $\epsilon \to 0$.

On (c), $p = se^{i\pi}$, $dp = dse^{i\pi}$, $\sqrt{p} = \sqrt{s}e^{i\pi/2} = i\sqrt{s}$

$$\begin{aligned}
\frac{1}{2\pi i}\int \frac{e^{px}}{\sqrt{p}}\,dp &= \frac{1}{2\pi i}\int_R^\epsilon \frac{e^{-sx}}{i\sqrt{s}}(-ds) \\
&= -\frac{1}{2\pi}\int_\epsilon^R \frac{e^{-sx}}{\sqrt{s}}\,ds
\end{aligned}$$

Similarly, on (e), $p = se^{-\pi}$, $dp = dse^{-i\pi}$, $\sqrt{p} = \sqrt{s}e^{-i\pi/2} = -i\sqrt{s}$

$$\begin{aligned}
\frac{1}{2\pi i}\int \frac{e^{px}}{\sqrt{p}}\,dp &= \frac{1}{2\pi i}\int_\epsilon^R \frac{e^{-sx}}{-i\sqrt{s}}(-ds) \\
&= -\frac{1}{2\pi}\int_\epsilon^R \frac{e^{-sx}}{\sqrt{s}}\,ds
\end{aligned}$$

Since the integrand is regular inside and on the contour C,

$$\frac{1}{2\pi i}\oint \frac{e^{px}}{\sqrt{p}}\,dp = 0$$

$$\frac{1}{2\pi i}\int_{c-i\sqrt{R^2-c^2}}^{c+i\sqrt{R^2-c^2}} \frac{e^{px}}{\sqrt{p}}\,dp - \frac{1}{\pi}\int_\epsilon^R \frac{e^{-sx}}{\sqrt{s}}\,ds + O(R^{-1/2}) + O(\sqrt{\epsilon}) = 0$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{px}}{\sqrt{p}}\,dp = \frac{1}{\pi}\int_0^\infty \frac{e^{-sx}}{\sqrt{s}}\,ds$$

If we set $u^2 = sx$ in the second integral, we have

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \frac{e^{-u^2}}{\sqrt{u^2/x}}\frac{2u\,du}{x} &= \frac{1}{\sqrt{x}}\left(\frac{2}{\pi}\int_0^\infty e^{-u^2}\,du\right)\\ &= \frac{1}{\sqrt{\pi x}}\end{aligned}$$

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{p}}\right) = \frac{1}{\sqrt{\pi x}}$$