Appendix 1
The convergence of Cauchy sequences in $\mathbb{R}$ and $\mathbb{C}$

In order to prove that Cauchy sequences in $\mathbb{R}$ and $\mathbb{C}$ converge, we need first to consider the distinguishing property of the real numbers.

The real numbers $\mathbb{R}$ can be visualised as points on the number line. They are distinguished from the rational numbers $\mathbb{Q}$ by the following property, which we take as a defining axiom:

Least upper bound axiom.

A non-empty set of real numbers which is bounded above has a least upper bound.

That is: If we have a non-empty set S of real numbers x such that, for some number K, $x \leq K$ for every $x \in S$, then **there is a real number** l such that

(i) $x \leq l$ for every $x \in S$;

(ii) If $x \leq k$ for every $x \in S$, then $l \leq k$.

This least upper bound is denoted by *lub* or *sup* (for supremum).

This property distinguishes the real numbers from the rational numbers.

For example, the set of numbers

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

is a set of rational numbers.

Furthermore

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{i=0}^{n} \frac{n!}{(n-i)!i!}\left(\frac{1}{n}\right)^n \\
&= 1 + n\frac{1}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \dots \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots \\
&< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \\
&< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\
&< 3
\end{aligned}$$

so that these rational numbers are bounded above.

However this set does not have a **rational** least upper bound.

The least upper bound of this set, considered as real numbers, is e, which is not rational.

An immediate consequence of the least upper bound axiom is that a non-empty set of real numbers which is bounded below has a greatest lower bound (*glb*) or infimum (*inf*). (Consider the set consisting of $-x$ for every $x \in S$.)

Nested intervals.

Consider an infinite set of closed intervals

$$I_n = \{a_n \le x \le b_n\}$$

in $\mathbb{R}$ with the following two properties.

(i) $I_{n+1} \subset I_n$; i.e. $a_n \le a_{n+1} < b_{n+1} \le b_n$

(ii) $b_n - a_n \to 0$ as $n \to \infty$.

(We call these a set of **nested intervals**.)

Then there is precisely one real number l which lies in every interval.

(This result asserts both **existence** and **uniqueness**.)

Proof: Consider firstly the set $\{a_n\}$. This set is bounded above by b_n for any n.

Therefore this set has a least upper bound l_1 which has the two properties

$$a_n \le l_1 \le b_n \quad \text{for all } n\ .$$

The set $\{b_n\}$ is therefore bounded below by l_1, so that there is a greatest lower bound l_2 which has the properties

$$l_1 \le l_2 \le b_n \quad \text{for all } n.$$

Combining these results we see that

$$0 \le l_2 - l_1 \le b_n - a_n \quad \text{for all } n,$$

and since $b_n - a_n \to 0$ as $n \to \infty$, we must have $l_2 = l_1$.

This common limit is the required number.

Since $a_n \le l_1 \le b_n$ for each n, it lies in every interval.

To show uniqueness, consider any other number m.

If $m < l_1$, then m is not an upper bound for $\{a_n\}$, since l_1 is the **least** upper bound.

Therefore, $m < a_n$ for some n, and so m is not in the interval $I_n = [a_n, b_n]$.

Similarly, if $m > l_1$, $m > b_n$ for some n, and therefore there is an interval in which it does not lie. (In fact, once we have one such interval I_n, we know that it does not lie in any of the subsequent intervals I_{n+k} either.)

Therefore this common element is unique.

Comment. This property, that a set of nested intervals contains a unique real number is sometimes taken as the defining axiom for real numbers. In this case, the least upper bound axiom is then deduced as a consequence.

The Bolzano-Weierstrass Theorem.

A bounded infinite set of real numbers has a limit point.

Let S be our infinite set of numbers, and let a_1 be the greatest lower bound for S and b_1 be the least upper bound.

Divide the interval $[a_1, b_1]$ into two equal sections; $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$.

At least one of these sections contains infinitely many points of S.

If the first section contains infinitely many points of S, let $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$; otherwise let $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$.

In either case, $a_1 \le a_2 < b_2 \le b_1$, $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$ and there are infinitely many points of S in the interval $[a_2, b_2]$.

We proceed inductively to defined a nested sequence of intervals $[a_n, b_n]$ of length $\frac{1}{2^{n-1}}(b_1 - a_1)$, each of which contains infinitely many points of S.

Since the length of these intervals goes to 0 as $n \to \infty$, there is a unique number l which lies in every interval.

Given any $\epsilon > 0$, we can find an integer N such that $b_N - a_N < \epsilon$. For every point x in this interval, $|x - l| \leq b_N - a_N < \epsilon$, so that there are infinitely many points of S such that $|x - l| < \epsilon$. This shows that l is a limit point of S.

Note that a set may have more than one limit point.

For example. if

$$S = \left\{ (-1)^n \frac{n+1}{n} \right\} ,$$

then 1 and -1 are limit points of S, while if S is the set of rational numbers between 0 and 1 then **every** real number in $[0, 1]$ is a limit point!

We are now in a position to prove that a Cauchy sequence in $\mathbb{R}$ converges.

In order to apply the previous result, we need to show that such a sequence constitutes a bounded set.

Suppose that $\{x_n\}$ is a sequence of real numbers with the property that, given any $\epsilon > 0$, we can find an integer N such that $|x_n - x_m| < \epsilon$ for every $m, n > N$.

In particular, taking $\epsilon = 1$, $|x_n - x_m| < 1$ for all $m, n > N_1$ say.

Therefore $|x_n - x_{N_1+1}| < 1$ for every $n > N_1$, so that $|x_n| < 1 + |x_{N_1+1}|$ for every $n > N_1$, and the sequence is bounded.

In order to apply the Bolzano-Weierstrass theorem we also need an infinite set. In practice the terms of a sequence are usually different. However we need to guard against occasional pathalogical examples.

If the sequence takes only a finite number of different values for $n > N_1$, then there must be some value l which is taken infinitely often.

But now, given any $\epsilon > 0$ there is an integer N such that $|x_n - x_m| < \epsilon$ for all $n, m > N$. Since l occurs infinitely often in the sequence, one at least of these x_m takes the value l, so that $|x_n - l| < \epsilon$ for all $n > N$, and the sequence converges to l.

Otherwise, the sequence represents a bounded infinite set in $\mathbb{R}$, so that, by the Bozano-Weierstrass Theorem, the sequence has a limit point l.

Given any $\epsilon > 0$, there are infinitely many points of the sequence such that $|x_m - l| < \frac{1}{2}\epsilon$, and there is an integer N such that $|x_n - x_m| < \frac{1}{2}\epsilon$ for all $m, n > N$. Since at least one of these x_m coincide, it follows that $|x_n - l| < \epsilon$ for all $n > N$, and the sequence converges to l.

From this result it follows that a Cauchy sequence in $\mathbb{C}$ also converges.

Suppose that $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}$;
i.e. given any $\epsilon > 0$ there is an integer N such that $|z_n - z_m| < \epsilon \ \forall \ m.n > N$;
and set $z_n = x_n + iy_n$.

Since $|x_n - x_m| \leq |z_n - z_m|$, $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ which converges to some limit l_1.

Similarly $\{y_n\}$ is a Cauchy sequence in $\mathbb{R}$, converging to l_2 say.

Hence $\{z_n\}$ converges to $l_1 + il_2$.