The point form of quantum dynamics and a 4-vector coordinate operator for a spinless particle

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(Received 20 December 1977)

We construct the analog in the quantum mechanics of a free spinless particle, of Dirac's formula for the generators of space–time translations in his point form of classical dynamics, where one takes as fundamental variables the generators of homogeneous Lorentz transformations and the coordinate 4-vector of the point where the world line of the particle meets one sheet of a two-sheeted hyperboloid in space–time. A 4-vector coordinate operator is determined for such a particle, with commuting Hermitian components. The corresponding observable is the analog of the coordinate 4-vector of the point on the hyperboloid. This operator bears the same relation to such a surface as the Newton–Wigner operator does to an instant.

1. INTRODUCTION

Dirac[1] has shown that in classical mechanics there are many ways to set up a dynamical discription of a free point particle with nonzero rest mass, consistent with the requirements of the special theory of relativity. An observer in any inertial frame of reference need make use only of variables specifying the condition of the particle at the point in space—time where its world line crosses an arbitrarily chosen spacelike surface. (Even a surface on which every two points are separated by either a spacelike or a null interval may be chosen.) These dynamical variables will include the energy—momentum 4-vector for the particle, P_λ; the relativistic angular—momentum tensor $J_{\lambda\mu}\,(=-J_{\mu\lambda})$; and three coordinates specifying the location of the particle on the surface.

A Poisson bracket (A,B) must be introduced for every pair of dynamical variables A,B in such a way that

$$\begin{aligned}&(P_\lambda,P_\mu)=0,\quad (P_\lambda,J_{\mu\nu})=g_{\lambda\mu}P_\nu-g_{\lambda\nu}P_\mu,\\&(J_{\lambda\mu},J_{\nu\rho})=g_{\mu\nu}J_{\lambda\rho}+g_{\lambda\rho}J_{\mu\nu}-g_{\mu\rho}J_{\lambda\nu}-g_{\lambda\nu}J_{\mu\rho},\end{aligned}\tag{1}$$

in order that the ten variables P_λ and $J_{\lambda\mu}$ will generate a group of transformations isomorphic to $\mathcal{P}$, the inhomogeneous Lorentz group.

If the observer adopts a system of space—time coordinates x_λ, an obvious choice for the spacelike surface is an instant, say $x_0=0$. The three coordinates q_r for the particle may then be taken to form a 3-vector. Following Dirac, one may choose to regard this as the 3-vector part of a 4-vector q_λ associated with the particle, and subject to the constraint $q_0=0$; and one may further suppose the existence of a 4-vector p_λ conjugate to q_λ, so that one has the Poisson bracket relations

$$(q_\lambda,q_\mu)=0=(p_\lambda,p_\mu),\quad (p_\lambda,q_\mu)=g_{\lambda\mu}.\tag{2}$$

However, the constraint $q_0=0$ is required to be invariant under canonical transformations generated by all dynamical variables of physical importance—called *physical variables* by Dirac—and it follows that these can be functions only of the q_r and p_r. In particular, Dirac argued that P_λ and $J_{\lambda\mu}$ are given by

$$P_r=p_r,\quad J_{rs}=q_rp_s-q_sp_r,\tag{3a}$$

$$P_0=(p_rp_r+m^2c^2)^{1/2}\tag{3b}$$

$$J_{r0}=(p_sp_s+m^2c^2)^{1/2}q_r,\tag{3c}$$

where m is the rest mass of the particle, and c the speed of light. It is well known that these definitions lead, in consequence of (2), to the required relations (1), as well as to the relations

$$P_\lambda P^\lambda=m^2c^2,\tag{4a}$$

$$P_0\geqq mc,\tag{4b}$$

$$\epsilon_{\lambda\mu\nu\rho}J^{\mu\nu}P^\rho=0,\tag{4c}$$

which characterize a system with rest mass m, positive energy, and no internal angular momentum. As Dirac pointed out, the choice of the surface $x_0=0$ singles out the Euclidean subgroup of $\mathcal{P}$, because coordinate transformations in that subgroup leave this surface invariant. Some consequences of this are the relatively simple expressions for the associated generators P_r and J_{rs} in (3) as compared with those for P_0 and J_{r0}, and a complicated transformation law for q_r with respect to Lorentz boosts.[2]

Dirac called this the "instant" form of dynamics, and presented corresponding results for two other forms of classical dynamics:

(I) The "point" form, in which the selected surface is taken to be one sheet of a two-sheeted hyperboloid or cone, such as $x_\lambda x^\lambda=k^2$, $x_0\geqq k\geqq 0$.

(II) The "front" form, in which the selected surface is a plane light wave front, such as $x_0=x_3$.

We are concerned with the point form of relativistic quantum dynamics (in the case $k>0$) for a free, spinless particle with nonzero rest mass and positive energy, in particular as it bears upon the much-discussed question of the definition of position operators for such a particle. We do not consider the special limiting case $k=0$, which has been discussed from a slightly different viewpoint by Peres,[3] and when we speak of the point form below, we shall generally be referring to the case $k>0$.

In order to establish the line of our argument, we describe briefly in Sec. 2 the familiar instant form of quantum dynamics for such a particle. The operator analog of the classical coordinate 3-vector q_r is easily seen to be the Newton–Wigner operator[4], which therefore corresponds to the measurement of the position of

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the particle on an instant. The complicated transformation properties of this operator in respect of Lorentz boosts are from this point of view not to be regarded as a defect when this operator is used to define the concept of localization of the particle (on an instant). On the contrary, they are a necessary consequence of the fact that an instant is not a Lorentz-invariant surface, and they are quite analogous to the transformation properties of the classical 3-vector q_r. (See however the relevant discussion in Ref. 2.) When the papers of Dirac[1] and Newton and Wigner[4] are studied side by side, one's initial reaction may be that the question of localization of an elementary particle (or system) on an instant has been resolved, at least in the spinless case, in a perfectly satisfactory way with due regard for the correspondence principle. One may feel less sure when one remembers that "manifestly covariant" descriptions of particles apparently need to be used if a (field) theory of local interactions is to be developed. Associated with such descriptions one has conserved current densities, which seem to point the way to other concepts of localization. (See for example Barut and Malin[5].)

In Sec. 3 we review Dirac's formula for the generators of space—time translation in the point form of classical mechanics and formulate the problem of finding the analog of this formula in quantum mechanics. We find that a mathematically equivalent problem can also be formulated—that of finding in terms of the group generators an expression for a 4-vector operator which is the analog in quantum mechanics of the coordinate 4-vector of the point where the world line of the classical particle meets the hyperboloid sheet described in (I) above.

Some of the properties of this 4-vector coordinate operator have been summarized by us elsewhere[6] without proof. It has commuting, Hermitian components, and bears the same relation to the surface described in (I) as does the Newton—Wigner operator to the instant; and just as the latter transforms simply under the Euclidean group, but not under Lorentz boosts, so the former transforms simply under the homogeneous Lorentz group (as a 4-vector) but not simply under translations in space or time. No doubt this explains why it does not seem to have been mentioned in the extensive literature on the localization of elementary particles. (See for example Refs. 4, 5, 7-11 and references therein.)

In Sec. 4, we solve the problem formulated in Sec. 3, relegating some proofs to two Appendices. Our conclusions are summarized in Sec. 5.

Notation: In what follows, we use the same symbol to denote a classical variable and its operator counterpart in quantum mechanics, relying on context to distinguish the two. Greek indices run over 0, 1, 2, 3 and Latin over 1, 2, 3. The metric tensor is diagonal with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, and the alternating tensor is defined with $\epsilon_{0123} = 1$.

2. THE INSTANT FORM OF RELATIVISTIC QUANTUM DYNAMICS

We proceed by analogy with the development of the classical case in Eqs. (2)—(4) now taking p_λ and q_λ to be operators in a suitable vector space, and replacing the Poisson bracket relations (2) by commutation relations

$$[q_\lambda, q_\mu] = 0 = [p_\lambda, p_\mu], \quad [p_\lambda, q_\mu] = i\hbar g_{\lambda\mu} \tag{5}$$

The constraints $q_0 = 0$ must now be interpreted as an operator equation valid in the physically relevant portion of the vector space, and it again restricts the physical variables to be functions only of the p_r and q_r. In quantum mechanics one also requires that physical variables be Hermitian operators in Hilbert space. Thus q_r should be taken to be Hermitian, and the assumption that p_r also is Hermitian guarantees the hermiticity of P_r and J_{rs} if we take over the classical formulas (3a) to define these variables. Moreover, P_0 as defined in (3b) will then also be Hermitian provided the operator square root is suitably interpreted. However, the formula (3c) for J_{r0} is then not consistent with the hermiticity requirement, and in order to obtain Hermitian J_{r0} without violating the correspondence principle, we are naturally led to adopt instead the symmetrized expression

$$J_{r0} = \tfrac{1}{2} q_r (p_s p_s + m^2c^2)^{1/2} + \tfrac{1}{2}(p_s p_s + m^2c^2)^{1/2} q_r. \tag{6}$$

It is then readily checked that with these definitions, P_λ and $J_{\lambda\mu}$ satisfy the required relations

$$\begin{aligned} &[P_\lambda, P_\mu] = 0, \quad [P_\lambda, J_{\mu\nu}] = i\hbar(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu), \\ &[J_{\lambda\mu}, J_{\nu\rho}] = i\hbar(g_{\mu\nu}J_{\lambda\rho} + g_{\lambda\rho}J_{\mu\nu} - g_{\mu\rho}J_{\lambda\nu} - g_{\lambda\nu}J_{\mu\rho}), \end{aligned} \tag{7}$$

as well as the "representation relations" (4), now interpreted as operator equations.

As q_r and p_r are by assumption Hermitian operators satisfying the canonical commutation relations, one may take the Hilbert space to be that of square-integrable functions $\Phi(p_1, p_2, p_3)$ with scalar product

$$(\Phi, \Psi) = \int\int\int_{-\infty}^{\infty} \Phi^*(p_1, p_2, p_3)\Psi(p_1, p_2, p_3)\, d^3p, \tag{8}$$

and take $q_r = i\hbar\,\partial/\partial p_r$. It is well-known (see, for example, Foldy[12]) that in this space the operators P_λ and $J_{\lambda\mu}$ defined above generate the unitary representation $(m^2c^2, 0, +)$ of $\mathcal{P}$, apprpriate to the description in the Heisenberg picture of a positive-energy spinless particle with rest mass m. The operators q_r are seen to form the Newton—Wigner 3-vector position operator (at $x_0 = 0$), which is thus revealed as the analogue in quantum mechanics of the position vector of the classical particle where its world line meets an instant.

Since the operators P_λ and $J_{\lambda\mu}$ do generate an irreducible representation of $\mathcal{P}$ one may hope that any given operator on the Hilbert space can be expressed in terms of those generators. This is true for the operator q_r: It can be seen from (3a), (3b), and (6) that

$$q_r = J_{r0}(P_0)^{-1} + \tfrac{1}{2} i\hbar P_r (P_0)^{-2}. \tag{9}$$

Moreover it is not hard to see that if one takes (9) as a definition of q_r, *assuming* the commutation relations (7) *and* representation relations (4), then one can *deduce* the relations

$$\begin{aligned} &[q_r, q_s] = 0, \quad [q_r, P_s] = i\hbar\,\delta_{rs}, \quad J_{rs} = q_r P_s - q_s P_r, \\ &\text{and} \\ &J_{r0} = \tfrac{1}{2} q_r (P_s P_s + m^2c^2)^{1/2} + \tfrac{1}{2}(P_s P_s + m^2c^2)^{1/2} q_r \end{aligned} \tag{10}$$

From this point of view, it is the group generators which play the more fundamental role, the variables q_r

being derived quantities. In such a formulation one does not need to introduce the unphysical variables p_0 and q_0.

3. THE POINT FORM OF RELATIVISTIC DYNAMICS

The preceding discussion should have its counterpart for each of the other forms of classical dynamics described by Dirac— and in principle, for the form corresponding to an arbitrary choice of spacelike surface. Fleming[11] has found the counterpart for the case of a general spacelike hyperplane, in which case the constraint $q_0=0$ is replaced by

$$q_\mu \eta^\mu = \tau,$$

where τ and η^μ are real constants, with $\eta_\mu\eta^\mu=1$. In the case of a nonplanar surface, such as that defining the point form, the counterpart is more difficult to discover.

In the point form, the three coordinates specifying the point at which the world line of the classical particle meets the surface $x_\lambda x^\lambda = k^2$, $x_0 \geqq k \geqq 0$, may be written as a 4-vector q_λ subject to the constraints

$$q_\lambda q^\lambda = k^2, \quad q_0 \geqq k. \tag{11}$$

Again introducing variables p_λ conjugate to q_λ as in (2), and noting these constraints, we expect that in this case physical variables can be functions only of q_λ and $l_{\lambda\mu}$, where

$$l_{\lambda\mu} = q_\lambda p_\mu - q_\mu p_\lambda.$$

Dirac argued that P_λ and $J_{\lambda\mu}$ in particular have the forms

$$\begin{aligned} J_{\lambda\mu} &= l_{\lambda\mu} \\ k^2 P_\lambda &= k^2 p_\lambda + q_\lambda\{[(p_\mu q^\mu)^2 - k^2(p_\mu p^\mu - m^2c^2)]^{1/2} - p_\mu q^\mu\} \\ &= -\, l_{\lambda\mu} q^\mu + q_\lambda (m^2c^2k^2 - \tfrac{1}{2} l_{\mu\nu} l^{\mu\nu})^{1/2}, \end{aligned} \tag{12}$$

which one readily checks imply the required relations (1) and (4).

One then sees that

$$q_\lambda P_\mu - q_\mu P_\lambda = J_{\lambda\mu} \tag{13}$$

and that the Poisson bracket (q_λ, P_μ) $[= -\partial P_\mu / \partial p^\lambda]$ has the form

$$(q_\lambda, P_\mu) = -g_{\lambda\mu} + q_\lambda P_\mu (q_\nu P^\nu)^{-1}, \tag{14}$$

so that

$$(q_\lambda, P^\lambda) = -3. \tag{15}$$

There is a remarkable symmetry between the roles of the timelike 4-vectors P_λ and q_λ in this form of dynamics. Apart from Eqs. (13— 15), ones sees that as a result of Eqs. (4a), (11), (12), and (13)

$$k^2 P_\lambda = -J_{\lambda\mu} q^\mu + q_\lambda (m^2c^2k^2 - \tfrac{1}{2} J_{\mu\nu} J^{\mu\nu})^{1/2} \tag{16}$$

$$m^2c^2 q_\lambda = J_{\lambda\mu} P^\mu + P_\lambda (m^2c^2k^2 - \tfrac{1}{2} J_{\mu\nu} J^{\mu\nu})^{1/2} \tag{17}$$

All equations in the point form of dynamics, such as (13)— (17) above, can be expressed in a manifestly covariant way using four-dimensional tensor notation. As Dirac stressed, this reflects the special role chosen for the homogeneous Lorentz subgroup of $\mathcal{P}$ by the observer's choice of a special surface which is invariant under Lorentz transformations of his coordinate system. It is this feature of the point form, and the associated fact that this subgroup is simple in the mathematical sense, which makes it attractive and will be responsible for any advantages it may have over the other forms.

Turning now to the point form of *quantum*dynamics for a spinless particle, we proceed initially as in the case of the instant form (Sec. 2), introduce a set of operators q_λ, p_λ satisfying the commutation relations (5), and impose the operator constraints

$$q_\lambda q^\lambda = k^2, \quad q_0 \geqq k > 0, \tag{18}$$

now to be satisfied by Hermitean operators q_λ on a Hilbert space $\mathcal{H}$.

We also take $l_{\lambda\mu} (= q_\lambda p_\mu - q_\mu p_\lambda)$ to be Hermitian operators in $\mathcal{H}$, and identify $J_{\lambda\mu} = l_{\lambda\mu}$. Then we wish to find the analog in the quantum theory of the formula (16), defining Hermitean P_λ in terms of q_λ and $J_{\lambda\mu}$ in such a way that the P_λ and $J_{\lambda\mu}$ satisfy the relations (4) and (7).

A suitable realization of the space $\mathcal{H}$ is that of functions $\Phi(q_0, q_1, q_2, q_3)$ defined on the sheet $q_0 \geqq k > 0$ of the hyperboloid $q_\lambda q^\lambda = k^2$, with scalar product

$$(\Phi, \Psi) = \int \Phi^*(q_0, q_1, q_2, q_3)\, \Psi(q_0, q_1, q_2, q_3)\, d\Omega,$$

where the integral is over the whole sheet, and $d\Omega (= dq_1 dq_2 dq_3 / q_0)$ is the Lorentz-invariant volume element on the sheet. Then q_λ and also $l_{\lambda\mu}$, which is now given by

$$l_{\lambda\mu} = i\hbar (q_\lambda \partial/\partial q^\mu - q_\mu \partial/\partial q^\lambda), \tag{19}$$

are Hermitian as required. In these terms, the problem is to exhibit the unitary representation $(m^2c^2, 0, +)$ of $\mathcal{P}$ in this function space, with generators

$$J_{\lambda\mu} = i\hbar (q_\lambda \partial/\partial q^\mu - q_\mu \partial/\partial q^\lambda) \tag{20}$$

and P_λ, defined in such a way that the classical formula (13), and consequently (16), can be recovered in the classical limit.

Peres[3] has solved the corresponding problem in the limiting case $k=0$ (for particles with spin 0 or $\frac{1}{2}$.) However, his solution is not expressed in a manifestly covariant way, so that the peculiar advantage of the point form is to some extent lost in his treatment. While Fubini, Hanson, and Jackiw,[13] Sommerfield,[14] and Gromes, Rothe, and Stech[15] have considered the initial-value problem for quantum or classical fields with the surface $x_\lambda x^\lambda = k^2$, $x_0 \geqq k > 0$ as Cauchy surface, no one to our knowledge has tackled the specific problem posed above, although Fubini *et al.* make passing reference to its difficulty.

Supposing that a solution P_λ exists, it is clear that $q_\lambda P_\mu - q_\mu P_\lambda$ and $P_\mu q_\lambda - P_\lambda q_\mu$ can differ from $J_{\lambda\mu}$ only by terms which in some sense vanish in the classical limit. We shall see that it is possible to find a solution P_λ with

$$J_{\lambda\mu} = \tfrac{1}{2}(q_\lambda P_\mu - q_\mu P_\lambda) + \tfrac{1}{2}(P_\mu q_\lambda - P_\lambda q_\mu),$$

but *not* with $J_{\lambda\mu}$ equal to either of the asymmetrical forms. The solution is uniquely determined if we require in addition the analogue of (15), viz.

$$[q_\lambda, P^\lambda] = -3i\hbar.$$

In the representation space $\mathcal{H}$ the P_λ and $J_{\lambda\mu}$ must satisfy the representation relations (4) as well as the commutation relations (7). We note that in $\mathcal{H}$ we shall have the relations (18), and also

$$\epsilon_{\lambda\mu\nu\rho}J^{\mu\nu}q^{\rho}=0 \tag{21}$$

by virtue of (2). Furthermore, it is clear that in $\mathcal{H}$

$$[q_\lambda, q_\mu]=0$$
$$[q_\lambda, J_{\mu\nu}]=i\hbar(g_{\lambda\mu}q_\nu - g_{\lambda\nu}q_\mu). \tag{22}$$

Comparing (18) and (21) with (4a)— (4c), and (22) with (7), we see that the Hermitian operators q_λ and $J_{\lambda\mu}$ may be regarded as generators in $\mathcal{H}$ of a unitary irreducible representation $(k^2, 0, +)$ of a group isomorphic to $\mathcal{P}$.

Just as in the case of the instant form of dynamics, one can look at the problem posed above for the point form from a different point of view, supposing the generators $P_\lambda, J_{\lambda\mu}$ of the representation $(m^2c^2, 0, +)$ to be given, and arguing that other operators in the Hilbert space, such as q_λ, might be expressible in terms of those generators. From this point of view, one has Hermitian operators P_λ, $J_{\lambda\mu}$ satisfying (4, 7), and wishes to express in terms of them, Hermitian operators q_λ via a formula reducing to (17) in the classical limit.

One sees that mathematically, this problem is essentially the same as the former one. In its most abstract form, *each* problem can be formulated as follows:

One has a set of Hermitian operators a_λ, $J_{\lambda\mu}$ which generate in a Hilbert space $\mathcal{H}$ a unitary irreducible representation $(1, 0, +)$ of a group isomorphic to $\mathcal{P}$, and hence satisfy

$$[a_\lambda, a_\mu]=0, \quad [a_\lambda, J_{\mu\nu}]=i\hbar(g_{\lambda\mu}a_\nu - g_{\lambda\nu}a_\mu), \tag{23a, b}$$

$$[J_{\lambda\mu}, J_{\nu\rho}]=i\hbar(g_{\mu\nu}J_{\lambda\rho}+g_{\lambda\rho}J_{\mu\nu}-g_{\lambda\nu}J_{\mu\rho}-g_{\mu\rho}J_{\lambda\nu}), \tag{23c}$$

as well as

$$a_\lambda a^\lambda = 1, \quad a_0 \geqq 1, \quad \epsilon_{\lambda\mu\nu\rho}J^{\mu\nu}a^\rho = 0; \tag{24a, b, c}$$

and one wishes to find, in terms of a_λ and $J_{\lambda\mu}$, Hermitian operators b_λ satisfying the same relations as a_λ in (23) and (24). In addition, one wants to obtain in the classical limit

$$a_\lambda b_\mu - a_\mu b_\lambda = \alpha J_{\lambda\mu} \tag{25}$$

and (consequently)

$$b_\lambda = -\alpha J_{\lambda\mu}a^\mu + a_\lambda(1-\tfrac{1}{2}\alpha^2 J_{\mu\nu}J^{\mu\nu})^{1/2}, \tag{26}$$

where α is a nonzero constant with the dimensions of $\hbar^{-1}$. In the first problem posed, $\alpha = 1/mck$, $a_\lambda = q_\lambda/k$, $b_\lambda = P_\lambda/mc$; in the second $\alpha = -1/mck$, $a_\lambda = P_\lambda/mc$, $b_\lambda = q_\lambda/k$.

As indicated above, in order to specify a solution uniquely, we shall find it necessary to require

$$[a_\lambda, b^\lambda] = -3i\hbar\alpha, \tag{27}$$

the analog of the classical relation $(a_\lambda, b^\lambda) = -3\alpha$.

4. SOLUTION OF THE PROBLEM

It is known[16] that the operator $\frac{1}{2}J_{\lambda\mu}J^{\lambda\mu}$ has in $\mathcal{H}$ a continuous spectrum of points $-(1+\beta^2)\hbar^2$, $0 \leqq \beta < \infty$; we introduce an Hermitian Lorentz-scalar operator B satisfying

$$[B, J_{\lambda\mu}]=0, \quad \tfrac{1}{2}J_{\lambda\mu}J^{\lambda\mu} = -(1+B^2)\hbar^2 \tag{28a, b}$$

(Here and below, a numerical multiple of the identity operator on $\mathcal{H}$ is denoted by the corresponding complex number.) The specification of B is completed by the requirement that its spectrum be of points β, $0 \leqq \beta < \infty$, corresponding to that of $\frac{1}{2}J_{\lambda\mu}J^{\lambda\mu}$ in the obvious way. Then $\hbar B$ is an analog in quantum theory of the positive square root of the positive classical quantity $(-\frac{1}{2}J_{\lambda\mu}J^{\lambda\mu})$.

The identity

$$J_{\lambda\mu}J^{\mu\nu}a_\nu - 2i\hbar J_{\lambda\nu}a^\nu + (\tfrac{1}{2}J_{\rho\sigma}J^{\rho\sigma})\,a_\lambda = 0 \tag{29}$$

holds as a consequence of (23c) and (24c), as can be checked by substitution of index values. This is a special case of a more general type of identity, discussed in detail elsewhere.[6,17]

Combining (28b) and (29) we have

$$(J_{\lambda\mu} - (i-B)\,\hbar g_{\lambda\mu})\,(J^{\mu\nu} - (i+B)\,\hbar g^{\mu\nu})\,a_\nu = 0. \tag{30}$$

We define

$$v^{(\pm)}{}_\lambda = (2\hbar B)^{-1}(J_{\lambda\mu} - (i \mp B)\,\hbar g_{\lambda\mu})a^\mu \tag{31}$$

so that

$$a_\lambda = v^{(+)}{}_\lambda - v^{(-)}{}_\lambda. \tag{32}$$

Then it follows from (30) that

$$J_{\lambda\mu}v^{(\pm)\mu} = \hbar(i \pm B)v^{(\pm)}{}_\lambda. \tag{33}$$

Now according to (31), $v^{(\pm)}{}_\lambda$ is a 4-vector operator, so that

$$[v^{(\pm)}{}_\lambda, J_{\mu\nu}] = i\hbar(g_{\lambda\mu}v^{(\pm)}{}_\nu - g_{\lambda\nu}v^{(\pm)}{}_\mu) \tag{34}$$

and hence

$$[v^{(\pm)}{}_\lambda, \tfrac{1}{2}J_{\mu\nu}J^{\mu\nu}] = 2i\hbar J_{\lambda\mu}v^{(\pm)\mu} + 3\hbar^2 v^{(\pm)}{}_\lambda. \tag{35}$$

Combining (28b), (33), and (35) we have

$$[v^{(\pm)}{}_\lambda, (1+B^2)] = (-1 \mp 2iB)v^{(\pm)}{}_\lambda,$$

that is

$$(B \mp i)^2 v^{(\pm)}{}_\lambda = v^{(\pm)}{}_\lambda B^2. \tag{36}$$

In Appendix A we show the validity of the (apparently) stronger result

$$Bv^{(\pm)}{}_\lambda = v^{(\pm)}{}_\lambda(B \pm i), \tag{37}$$

and in Appendix B that (37) implies with (32), the hermiticity of $v^{(\pm)}{}_\lambda$. It must be emphasized that (37) is *consistent* with the hermiticity of $v^{(\pm)}{}_\lambda$ and B, although formally it seems to imply that $v^{(\pm)}{}_\lambda$ shifts an eigenvector of B corresponding to a real eigenvalue β, to one corresponding to the complex eigenvalue $(\beta \pm i)$. The point is that B has no eigenvectors in $\mathcal{H}$, and *a fortiori*, none in the domain of $v^{(\pm)}{}_\lambda$. [A similar situation occurs in those unitary representations of the conformal group corresponding to massless particles, where the Hermitian dilatation generator D and translation generators P_λ satisfy $DP_\lambda = P_\lambda(D-i)$.]

The operators $v^{(\pm)}{}_\lambda$ have several remarkable properties which we now list, putting derivations in Appendix B, and they play a central role in what follows. One has

$$v^{(\pm)}{}_\lambda{}^\dagger = v^{(\pm)}{}_\lambda \quad \text{(Hermiticity)} \tag{38a}$$

$$v^{(\pm)}{}_\lambda v^{(\pm)\lambda} = 0 \tag{38b}$$

$$v^{(\pm)}{}_\lambda v^{(\mp)\lambda} = -(B \mp i)/2B \tag{38c}$$

$$[v^{(\pm)}{}_\lambda, v^{(\pm)}{}_\mu] = 0 \tag{38d}$$

$$v^{(\pm)}{}_\lambda v^{(\mp)}{}_\mu - v^{(\pm)}{}_\mu v^{(\mp)}{}_\lambda = \mp J_{\lambda\mu}/2\hbar B. \tag{38e}$$

We now tackle the problem posed in Sec. 3, seeking an expression for the operator b_λ in the form

$$b_\lambda = F^{(+)}(B)\, v^{(+)}{}_\lambda + F^{(-)}(B)\, v^{(-)}{}_\lambda, \tag{39}$$

where $F^{(\pm)}(B)$ is a Lorentz-scalar operator-valued function of B. We shall think of $F^{(\pm)}$ as an "analytic" function with, for example on suitable vectors in $\mathcal{H}$,

$$F^{(+)}(B) = \sum_{n=0}^{\infty} a_n (B-b)^n \tag{40}$$

for some complex numbers a_n and b. The Hermitian conjugate of $F^{(+)}$ is then

$$\sum_{n=0}^{\infty} a_n{}^*(B-b^*)^n,$$

which we write as $F^{(+)*}(B)$. Thus

$$F^{(\pm)}(B)^\dagger = F^{(\pm)*}(B). \tag{41}$$

In what follows, we repeatedly make use of relations like

$$F^{(\pm)}(B) v^{(+)}{}_\lambda = v^{(+)}{}_\lambda F^{(\pm)}(B+i),$$

which are taken to follow from (37) and (40).

Taking the Hermitian conjugate of (39), we have

$$\begin{aligned} b_\lambda{}^\dagger &= v^{(+)}{}_\lambda F^{(+)*}(B) + v^{(-)}{}_\lambda F^{(-)*}(B) \\ &= F^{(+)*}(B-i) v^{(+)}{}_\lambda + F^{(-)*}(B+i) v^{(-)}{}_\lambda \end{aligned}$$

so that b_λ is Hermitian if and only if

$$F^{(\pm)}(B) = F^{(\pm)*}(B \mp i). \tag{42}$$

Next we note from (39) that

$$\begin{aligned} b_\lambda b_\mu = {} & F^{(+)}(B) F^{(+)}(B-i) v^{(+)}{}_\lambda v^{(+)}{}_\mu \\ & + F^{(+)}(B) F^{(-)}(B-i) v^{(+)}{}_\lambda v^{(-)}{}_\mu \\ & + F^{(-)}(B) F^{(+)}(B+i) v^{(-)}{}_\lambda v^{(+)}{}_\mu \\ & + F^{(-)}(B) F^{(-)}(B+i) v^{(-)}{}_\lambda v^{(-)}{}_\mu . \end{aligned} \tag{43}$$

Then as a consequence of (38b, c) we have

$$\begin{aligned} b_\lambda b^\lambda = {} & -(B-i) F^{(+)}(B) F^{(-)}(B-i)/2B \\ & -(B+i) F^{(-)}(B) F^{(+)}(B+i)/2B, \end{aligned}$$

so that $b_\lambda b^\lambda = 1$ if and only if

$$\begin{aligned} &(B-i) F^{(+)}(B) F^{(-)}(B-i) + (B+i) F^{(+)}(B+i) F^{(-)}(B) \\ &= -2B. \end{aligned} \tag{44}$$

Furthermore, (43) also implies, with (38d, e), that

$$\begin{aligned} &[b_\lambda, b_\mu] \\ &= (-F^{(+)}(B) F^{(-)}(B-i) + F^{(+)}(B+i) F^{(-)}(B)) J_{\lambda\mu}/2\hbar B, \end{aligned}$$

so that $[b_\lambda, b_\mu] = 0$ if and only if

$$F^{(+)}(B) F^{(-)}(B-i) = F^{(+)}(B+i) F^{(-)}(B). \tag{45}$$

Combining (44) and (45) we have at once

$$F^{(+)}(B) F^{(-)}(B-i) = -1. \tag{46}$$

Thus a four-vector operator of the form (39) is Hermitian, with commuting components and unit length, if and only if (42) and (46) hold. [In particular this is so for a_λ itself, for which $F^{(+)}(B) = -F^{(-)}(B) = 1$.]

Turning now to the analog of the classical formula (25), we might try to impose any of

$$a_\lambda b_\mu - a_\mu b_\lambda = \alpha J_{\lambda\mu}, \tag{47a}$$

$$b_\mu a_\lambda - b_\lambda a_\mu = \alpha J_{\lambda\mu}, \tag{47b}$$

$$(a_\lambda b_\mu - a_\mu b_\lambda) + (b_\mu a_\lambda - b_\lambda a_\mu) = 2\alpha J_{\lambda\mu}. \tag{47c}$$

Considering firstly (47a), we note that

$$\begin{aligned} a_\lambda b_\mu &= (v^{(+)}{}_\lambda - v^{(-)}{}_\lambda)(F^{(+)}(B) v^{(+)}{}_\mu + F^{(-)}(B) v^{(-)}{}_\mu) \\ &= F^{(+)}(B-i) v^{(+)}{}_\lambda v^{(+)}{}_\mu + F^{(-)}(B-i) v^{(+)}{}_\lambda v^{(-)}{}_\mu \\ &\quad - F^{(+)}(B+i) v^{(-)}{}_\lambda v^{(+)}{}_\mu - F^{(-)}(B+i) v^{(-)}{}_\lambda v^{(-)}{}_\mu, \end{aligned}$$

so that in view of (38d, e)

$$a_\lambda b_\mu - a_\mu b_\lambda = (-F^{(-)}(B-i) - F^{(+)}(B+i)) J_{\lambda\mu}/2\hbar B. \tag{48}$$

Thus (47a) holds if and only if

$$F^{(+)}(B+i) + F^{(-)}(B-i) = -2\alpha\hbar B. \tag{49}$$

Multiplying both sides of this equation by $F^{(+)}(B)$, and noting (46) we have

$$F^{(+)}(B) F^{(+)}(B+i) + 2\alpha\hbar B F^{(+)}(B) = 1. \tag{50}$$

Taking the Hermitian conjugate of (50), we get

$$F^{(+)*}(B)\, F^{(+)*}(B-i) + 2\alpha\hbar B F^{(+)*}(B) = 1.$$

which, in view of (42), is equivalent to

$$F^{(+)}(B+i) F^{(+)}(B) + 2\alpha\hbar B F^{(+)}(B+i) = 1. \tag{51}$$

Then (50) and (51) imply

$$F^{(+)}(B) = F^{(+)}(B+i). \tag{52}$$

But then (50) and (52) imply

$$F^{(+)}(B) F^{(+)}(B) + 2\alpha\hbar B F^{(+)}(B) = 1 \tag{53}$$

and

$$F^{(+)}(B+i) F^{(+)}(B+i) + 2\alpha\hbar B F^{(+)}(B+i) = 1; \tag{54}$$

and (53) implies

$$F^{(+)}(B+i) F^{(+)}(B+i) + 2\alpha\hbar (B+i) F^{(+)}(B+i) = 1,$$

which is inconsistent with (54). Thus (47a) is inconsistent with (42) and (46).

The attempt to impose (47b) rather than (47a) leads to similar conclusions, and we turn now to the symmetric case (47c). Just as we derived (48), we find also that

$$b_\mu a_\lambda - b_\lambda a_\mu = -(F^{(+)}(B) + F^{(-)}(B)) J_{\lambda\mu}/2\hbar B,$$

so that (47c) holds if and only if

$$[F^{(-)}(B-i) + F^{(+)}(B+i) + F^{(+)}(B) + F^{(-)}(B)] = -4\alpha\hbar B, \tag{55}$$

or equivalently,

$$G(B) + G(B+i) = -2\alpha\hbar B, \tag{56}$$

where

$$2G(B) = F^{(+)}(B) + F^{(-)}(B-i). \tag{57}$$

Taking the Hermitian conjugate of (57) and using (42), we get

$$G^*(B) = G(B+i). \tag{58}$$

Multiplying by $F^{(+)}(B)$ in (57) and using (46) we get

$$F^{(+)}(B)F^{(+)}(B) - 2G(B)F^{(+)}(B) - 1 = 0,$$

so that

$$F^{(+)}(B) = G(B) + [1 + G^2(B)]^{1/2}, \tag{59}$$

where $[1+G^2(B)]^{1/2}$ denotes some Lorentz-scalar square root of $[1+G^2(B)]$.

Then (46) implies that

$$F^{(-)}(B-i) = G(B) - [1+G^2(B)]^{1/2},$$

that is

$$F^{(-)}(B) = G(B+i) - [1 + G^2(B+i)]^{1/2}. \tag{60}$$

Conversely, if $G(B)$ satisfies (56) and (58), then $F^{(+)}(B)$ and $F^{(-)}(B)$, defined as in (59), (60), can be seen to satisfy (42), (46), and (55), and so to define via (39) a 4-vector operator b_λ with the required properties of hermiticity, unit length, commuting components, and satisfaction of (47c). Since (56) and (58) evidently do not uniquely determine $G(B)$, the solution to the problem at hand is not yet specified completely.

In order to remove this ambiguity, we now impose (27). From (32) and (39), we get with the help of (38b, c),

$$a_\lambda b^\lambda = [(B+i)F^{(+)}(B+i) - (B-i)F^{(-)}(B-i)]/2B$$

and

$$b_\lambda a^\lambda = [(B-i)F^{(+)}(B) - (B+i)F^{(-)}(B)]/2B,$$

so that (27) implies

$$(B+i)G(B+i) - (B-i)G(B) = -3i\alpha\hbar B. \tag{61}$$

Then (56) and (61) imply

$$G(B) = -\alpha\hbar(B - \tfrac{1}{2}i),$$

which is seen to be consistent with (58).

Substituting in (59), (60), we arrive at the expressions

$$F^{(\pm)}(B) = -\alpha\hbar(B \mp \tfrac{1}{2}i) \pm H^{(\pm)}(B), \tag{62}$$

where

$$H^{(\pm)}(B) = [1 + \alpha^2\hbar^2(B \mp \tfrac{1}{2}i)^2]^{1/2}. \tag{63}$$

Substituting from (62) in (39), and recalling the definition (31) of $v^{(\pm)}{}_\lambda$, we have finally

$$b_\lambda = [-\alpha + (H^{(+)}(B) - H^{(-)}(B))/2\hbar B]J_{\lambda\mu}a^\mu$$
$$+ [(3i\alpha\hbar B + (B-i)H^{(+)}(B) + (B+i)H^{(-)}(B))/2B]a_\lambda. \tag{64}$$

There is a remaining ambiguity concerning the definition of the operator square roots $H^{(\pm)}(B)$ in (63) and (64). This reflects the fact that we have yet to impose the condition $b_0 \geqq 1$. An Hermitian 4-vector operator with unit length, commuting components, and satisfying (47c) can have either $b_0 \geqq 1$ or $b_0 \leqq -1$. The second possibility corresponds to a classical formula like (26) with a minus sign preceding the a_λ therein

We can guarantee $b_0 \geqq 1$ by choosing the square roots $H^{(\pm)}(B)$ so that (64) will yield (26) in the classical limit. In that limit, it is clear that in some sense which we need not make precise,

$$\hbar \sim 0, \quad B \sim +\infty, \quad \hbar^2B^2 \sim -\tfrac{1}{2}J^{\mu\nu}.$$

Then

$$H^{(+)}(B) \sim H^{(-)}(B) \sim [1 - \tfrac{1}{2}\alpha^2 J_{\mu\nu}J^{\mu\nu}]^{1/2}$$

and (26) is recovered from (64), *provided* the point $H^{(\pm)}(\beta)$ in the spectrum of $H^{(\pm)}(B)$, corresponding to the point β in the spectrum of B, has positive real part $(\sim |\alpha|\hbar\beta)$ when $\beta \gg 0$. This is achieved by taking $H^{(\pm)}(\beta)$ to be the principal square root of $1 + \alpha^2\hbar^2(\beta \mp \frac{1}{2}i)^2$.

In other words, in the representation described in Appendix A, where B is realized as multiplication by the nonnegative quantity β, $H^{(\pm)}(B)$ will be realized as multiplication by the principal square root $[1 + \alpha^2\hbar^2(\beta \mp \frac{1}{2}i)^2]^{1/2}$. In this connection, it is worth mentioning that for $\alpha^2\hbar^2 \geqq 4$, the two curves $z^{(\pm)}(\beta) = [1 + \alpha^2\hbar^2(\beta \mp \frac{1}{2}i)^2]$, $\beta \geqq 0$, in the complex z plane meet the branch cut of the principal square root $z^{1/2}$ at $\beta = 0$. It *may* be that this means that b_λ as defined in (64) is not self-adjoint in such cases, but with our algebraic approach we cannot determine this. Note that $\alpha^2\hbar^2 < 4$ can be written in the very suggestive form

$$(mc)k > \tfrac{1}{2}\hbar.$$

5. CONCLUSION

We have derived for a spinless particle with nonzero rest mass m and positive energy, the analog in quantum mechanics of Dirac's formula (16) defining the point form of classical dynamics. Substituting $\alpha = 1/mck$, $a_\lambda = q_\lambda/k$, $b_\lambda = P_\lambda/mc$ in (64) we have

$$k^2P_\lambda = [-1 + mck(H^{(+)}(B) - H^{(-)}(B))/2\hbar B]J_{\lambda\mu}q^\mu$$
$$+[(3i\hbar B + mck(B-i)H^{(+)}(B)$$
$$+mck(B+i)H^{(-)}(B))/2B]q_\lambda, \tag{65}$$

where

$$H^{(\pm)}(B) = [1 + \hbar^2(B \mp \tfrac{1}{2}i)^2/m^2c^2k^2]^{1/2}, \tag{66}$$

By the same means, we have derived a formula for the 4-vector coordinate operator q_λ in terms of the generators of the inhomogeneous Lorentz group $\mathcal{P}$. Substituting $\alpha = -1/mck$, $a_\lambda = P_\lambda/mc$, $b_\lambda = q_\lambda/k$ in (64) we have

$$m^2c^2q_\lambda = [1 + mck(H^{(+)}(B) - H^{(-)}(B))/2\hbar B]J_{\lambda\mu}P^\mu$$
$$+ [(-3i\hbar B + mck(B-i)H^{(+)}(B)$$
$$+ mck(B+i)H^{(-)}(B))/2B]P_\lambda, \tag{67}$$

with $H^{(\pm)}(B)$ as in (66)

This 4-vector operator satisfies $q_\lambda q^\lambda = k^2$, $q_0 \geqq k > 0$, and has commuting components which are Hermitian. We have made implicity in the text certain reasonable but unproven assumptions about domains of definition of operators, and we certainly cannot claim to have proved the stronger condition of self-adjointness of q_λ. In particular, we have suggested that there is some question as to the self-adjointness of q_λ when one does not have

$$(mc)k > \tfrac{1}{2}\hbar.$$

The operator q_λ is the analogue in quantum mechanics of the coordinate 4-vector of the point where the world line of the classical particle meets the positive sheet of a two-sheeted hyperboloid, and as such certainly qualifies to be called a coordinate operator. It bears the same relation to this surface as the Newton–Wigner 3-vector operator does to an instant. Only transformations in the Euclidean subgroup of $\mathcal{P}$ leave an instant invariant, and accordingly the Newton—Wigner operator transforms simply under this subgroup but not simply under Lorentz boosts. Similarly, the sheet of the hyperboloid is Lorentz invariant, and q_λ transforms simply under the homogeneous Lorentz group (as a 4-vector) but not simply under translations in space and time. In particular, the canonical relations

$$[q_r, P_s] = i\hbar\,\delta_{rs}$$

do *not* hold. We have so far been unable to evaluate the commutator $[q_\lambda, P_\mu]$ in terms of simpler expressions, in order to find the analog of the classical equation (14).

It could reasonably be argued that the formula (67) is so complicated that one cannot hope to manipulate readily or usefully with the operator q_λ. Our main purpose has been to indicate the existence of this 4-vector operator, which we have seen is defined by the conditions that it has Hermitian, commuting components, and satisfies

$$\begin{aligned} &q_\lambda q^\lambda = k^2, \quad q_0 \geqq k > 0, \\ &(q_\lambda P_\mu - q_\mu P_\lambda) + (P_\mu q_\lambda - P_\lambda q_\mu) = 2J_{\lambda\mu}, \\ &[q_\lambda, P^\lambda] = -3i\hbar. \end{aligned} \tag{68}$$

Given that, we hope it will prove possible to investigate its properties further (for example, to find the common generalized eigenvectors of its components in, say, the momentum representation) by other means.

ACKNOWLEDGMENTS

Helpful discussions with L. Bass and D.D. McGregor are gratefully acknowledged.

APPENDIX A

Chakrabarti, Levy-Nahas, and Seneor[18] have described the matrix elements of the operators a_λ in a "Lorentz basis" for the representation $(1, 0, +)$ of $\mathcal{P}$. This basis, in which B, $\frac{1}{2}J_{rs}J_{rs}$ and J_{12} are diagonalized is not a true basis, as it consists of nonnormalizeable vectors, but it can be used to define a realization of the underlying abstract structure. Then the Hilbert space is realized as the space of vectors

$$\phi = (\phi_{00}(\beta),\ \phi_{11}(\beta),\ \phi_{10}(\beta),\ \phi_{1-1}(\beta),\ \phi_{22}(\beta)\cdots), \quad 0 \leqq \beta < \infty,$$

where

$$\sum_{l,m} \left(\int_0^\infty |\phi_{lm}(\beta)|^2 d\beta\right) < \infty,$$

the sum being over $m = l,\ l-1, \ldots, -l$ for $l = 0, 1, 2\cdots$. The scalar product of two such vectors is

$$(\phi, \psi) = \sum_{l,m} \left(\int_0^\infty \phi^*_{lm}(\beta)\psi_{lm}(\beta)d\beta\right).$$

The results of Chakrabarti *et al.* [see in particular Eqs. (2.3), (2.18), (2.19), and (2.29) in Ref. 18] show that in this realization, the action of the operators a_0 and B is

$$\begin{aligned} (a_0\phi)_{lm}(\beta) &= [(l - i\beta)(l + i\beta + 1)/4\beta(\beta - i)]^{1/2}\phi_{lm}(\beta + i) \\ &\quad + [(l + i\beta)(l - i\beta + 1)/4\beta(\beta + i)]^{1/2}\phi_{lm}(\beta - i) \\ (B\phi)_{lm}(\beta) &= \beta\phi_{lm}(\beta). \end{aligned} \tag{A1}$$

Here the action of a_0 evidently presupposes certain analyticity properties of the functions $\phi_{lm}(\beta)$ for vectors ϕ in its domain. Although it is difficult to exhibit explicitly a common invariant dense domain of Hermiticity for a_0 and B, it is nevertheless clear from (A1) that

$$a_0 = u^{(+)}{}_0 - u^{(-)}{}_0 \tag{A2}$$

where, on suitable vectors ϕ

$$(u^{(\pm)}{}_0\phi)_{lm}(\beta) = \pm[(l \pm i\beta)(l \mp i\beta + 1)/4\beta(\beta \pm i)]^{1/2}\phi_{lm}(\beta \mp i)$$

and, consequently

$$Bu^{(\pm)}{}_0 = u^{(\pm)}{}_0(B \pm i). \tag{A3}$$

It follows from (A2) and (A3) that

$$u^{(\pm)}{}_0 = \pm\tfrac{1}{2}a_0 + \tfrac{1}{2}i[a_0, B]. \tag{A4}$$

Defining the 4-vector operator

$$u^{(\pm)}{}_\lambda = \pm\tfrac{1}{2}a_\lambda + \tfrac{1}{2}i[a_\lambda, B] \tag{A5}$$

so that

$$a_\lambda = u^{(+)}{}_\lambda - u^{(-)}{}_\lambda, \tag{A6}$$

we see from (A3) by covariance that

$$Bu^{(\pm)}{}_\lambda = u^{(\pm)}{}_\lambda(B \pm i), \tag{A7}$$

Since it then follows trivially that

$$(B \mp i)^2 u^{(\pm)}{}_\lambda = u^{(\pm)}{}_\lambda B^2 \tag{A8}$$

we see from (A6), (A8), (32), and (36) that $u^{(\pm)}{}_\lambda = v^{(\pm)}{}_\lambda$, and (37) is thus justified.

APPENDIX B

Once (37) is established, it follows using (32) that

$$v^{(\pm)}{}_\lambda = \pm\tfrac{1}{2}a_\lambda + \tfrac{1}{2}i[a_\lambda, B]. \tag{B1}$$

Since a_λ and B are Hermitian, it follows at once that $v^{(\pm)}{}_\lambda$ is also Hermitian; that is (38a) holds.

Now note from (B1) and (37) that

$$v^{(\pm)}{}_\lambda v^{(\pm)\lambda} = \pm\tfrac{1}{2}v^{(\pm)}{}_\lambda a^\lambda + \tfrac{1}{2}iv^{(\pm)}{}_\lambda a^\lambda B - \tfrac{1}{2}i(B \mp i)v^{(\pm)}{}_\lambda a^\lambda. \tag{B2}$$

But from (31)

$$\begin{aligned} v^{(\pm)}{}_\lambda a^\lambda &= (2\hbar B)^{-1}(J_{\lambda\mu}a^\mu a^\lambda - (i \mp B)\hbar a_\lambda a^\lambda) \\ &= -(i \mp B)/2B, \end{aligned} \tag{B3}$$

since the components of a_λ commute, and $a_\lambda a^\lambda = 1$. Combining (B2) and (B3) we have

$$v^{(\pm)}{}_\lambda v^{(\pm)\lambda} = 0,$$

establishing (38b). In a similar way, (38c) is deduced.

Again using (B1) and (37) we have

$$v^{(\pm)}{}_\lambda v^{(\pm)}{}_\mu = \pm\tfrac{1}{2}v^{(\pm)}{}_\lambda a_\mu + \tfrac{1}{2}iv^{(\pm)}{}_\lambda a_\mu B - \tfrac{1}{2}i(B \pm i)v^{(\pm)}{}_\lambda a_\mu. \tag{B4}$$

But from (31)

$$v^{(\pm)}{}_\lambda a_\mu = (2\hbar B)^{-1}(J_{\lambda\nu}a^\nu a_\mu - (i \mp B)\hbar a_\lambda a_\mu), \tag{B5}$$

so that

$$v^{(\pm)}{}_\lambda a_\mu - v^{(\pm)}{}_\mu a_\lambda = (2\hbar B)^{-1}(J_{\lambda\nu}a^\nu a_\mu - J_{\mu\nu}a^\nu a_\lambda). \tag{B6}$$

Now (24c) is equivalent to

$$J_{\lambda\nu}a_\mu + J_{\mu\lambda}a_\nu + J_{\nu\mu}a_\lambda = 0.$$

Contracting with a^ν from the right in this equation, and noting that $a_\nu a^\nu = 1$, we obtain

$$J_{\lambda\nu} a^\nu a_\mu - J_{\mu\nu} a^\nu a_\lambda = J_{\lambda\mu}. \tag{B7}$$

Combining (B6) and (B7) we have

$$v^{(\pm)}{}_\lambda a_\mu - v^{(\pm)}{}_\mu a_\lambda = (2\hbar B)^{-1} J_{\lambda\mu},$$

and thus from (B4) that

$$v^{(\pm)}{}_\lambda v^{(\pm)}{}_\mu - v^{(\pm)}{}_\mu v^{(\pm)}{}_\lambda = 0,$$

establishing (38d). In a similar way, (38e) is deduced.

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