Honours Thesis

Classical Randomness in a Quantum Walk On a Line

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Abstract

Recent success in the formulation of Quantum Random Walks, has motivated consideration of a quantum walk partially influenced by some classically random parameter. In this thesis, a classical random walk and a quantum random walk are firstly constructed and their contrasting properties investigated. Such analysis is fundamental to the creation of a hybrid process of the two walks. Numerical data indicates that the introduction of some classical randomness into a quantum random walk will produce classical-like results. This suggests a form of quantum Brownian Motion. Provided the number of iterations is large enough, this phenomenon appears to occur regardless of the 'degree of randomness' applied to the quantum system. Such numerical findings prompt the need for further analytic development and practical exploration in this field.

Chapter 4

'Random' Quantum Walk

Previous work has been done by Professor Bracken on exploring a hybrid of the classical and quantum random walks. His research has examined the influence that random behaviour has on the comparative processes: That is, how the addition of classical randomness effects the quantum system.

For the classical walk we began with a point at the origin. At each iteration in the walk we imagined that the point was 'rotated' either left or right randomly and then translated one step in that direction. For our 'pure' quantum walk, however, we began with a point at the origin, which moved in a superposition of left and right directions. At each iteration we rotated the system by appling a fixed rotation operator, and followed this with a suitable translation. The most effective way to produce a hybrid of these two process as required, then, is clearly to insinuate a degree of classical randomness into the quantum rotation operator.

The processes involved in the 'Random' Quantum Walk are the same as those for the 'pure' quantum walk, save the addition of the classically random element into the rotation operator, R. This chapter will numerically and analytically investigate the progression of the

'random' quantum walk and its resultant properties, in comparison to the Classical and 'pure' quantum walks.

4.1 Quantum Walk Formalism

A 'walk' requires a series of rotations and translation. A quantum walk achieves this through application of operators that act on the system. At each iteration, an operator V, acts on the position distribution to achieve a quantum walk; that is, rotations and translations.

$$V = (T^{(+)} + T^{(-)})R,$$

where $T_{\pm}$ are the translation operators and R is the rotation operator.

In the purely quantum case, the rotation and translation are effected by a fixed operator. That is, R can be arbitrarily chosen such that a suitable fixed rotation matrix is formed. For example,

$$R = \exp(i\frac{\pi}{4}\sigma_2) \quad = \quad \frac{1}{\sqrt{2}}(I + i\sigma_2)$$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

as previously given.

See Appendix (1) for a definition of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$.

The translation of the resulting position vector is achieved accordingly.

$$T^{(\pm)} = \exp(\mp\Delta\frac{\partial}{\partial x}) \ \frac{1}{2}(I_2 \pm \sigma_3)$$

The 'random' quantum walk is formulated similarly, but with a classically random element insinuated into the 'pure' rotation operator R, as previously stated.

For example, the rotation matrix could become

$$R_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & r_n \\ -r_n & 1 \end{pmatrix}$$

where r_n is randomly assigned ± 1 at each iteration independently, with equal probability.

These two alternative matrices act in opposing fashions on the position vector. Thus such a choice of rotation operator is more analogous to the classical walk than its 'pure' quantum counterpart, making it a relevant hybrid of the classical and quantum walks. As will be investigated in the following chapter, however, the choice of 'random' rotation operator is largely arbitrary.

Therefore, the walk progression operator, V, for the random quantum system is,

$$V = (T^{(+)} + T^{(-)})R_n,$$

now with

$$R_n = \exp(ir_n \frac{\pi}{4}\sigma_2) = \frac{1}{\sqrt{2}}(I + ir_n\sigma_2)$$

$$R_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & r_n \\ -r_n & 1 \end{pmatrix}$$

So that,

$$R_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} & r_n = +1 \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & r_n = -1 \end{cases}$$

as stated above.

4.2 Properties of the 'Random' Quantum Walk

4.2.1 Probability

The position matrix, which is acted on by the walk progression operator, V, does not measure the exact position of the point on a line, but rather a superposition of possible and probable states in which it could exist. The probability distribution after n iterations/steps, then, is determined as previously defined:

$$\psi_k^{(n)} = \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

$$\text{Probability,} \quad p_k = |u_k|^2 + |v_k|^2$$

The appropriate probability measures can be plotted against their placement in the vector, to form an approximate version of a quantum probability density function.

Now recall that the classical probability distribution is that of a Gaussian, whereas the purely quantum case displays a graph of opposing form.

The new case considered here is posed in the quantum regime but can be compared to the classical in its use of random rotation. After n=100 iterations the pdf of the system has the following form, averaged over 100 repetitions.

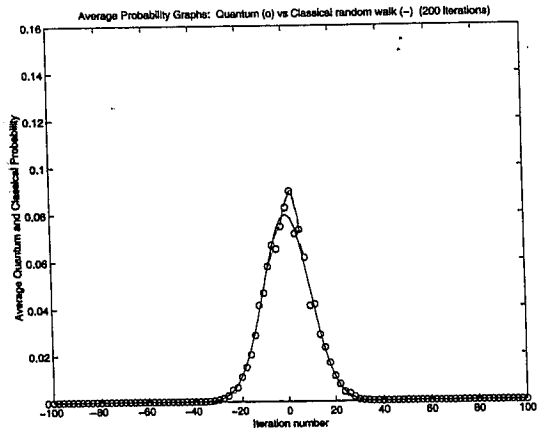

Figure 11: Classical versus Random Quantum pdfs

It can be posed that the resulting 'random' quantum pdf forms a bell-like shape, similar to that of the classical random walk. However, we must also investigate if it progresses as the classical walk does, before any conclusions can be made. Observe the evolution of the 'random' quantum walk pdf as the final time iteration is increased.

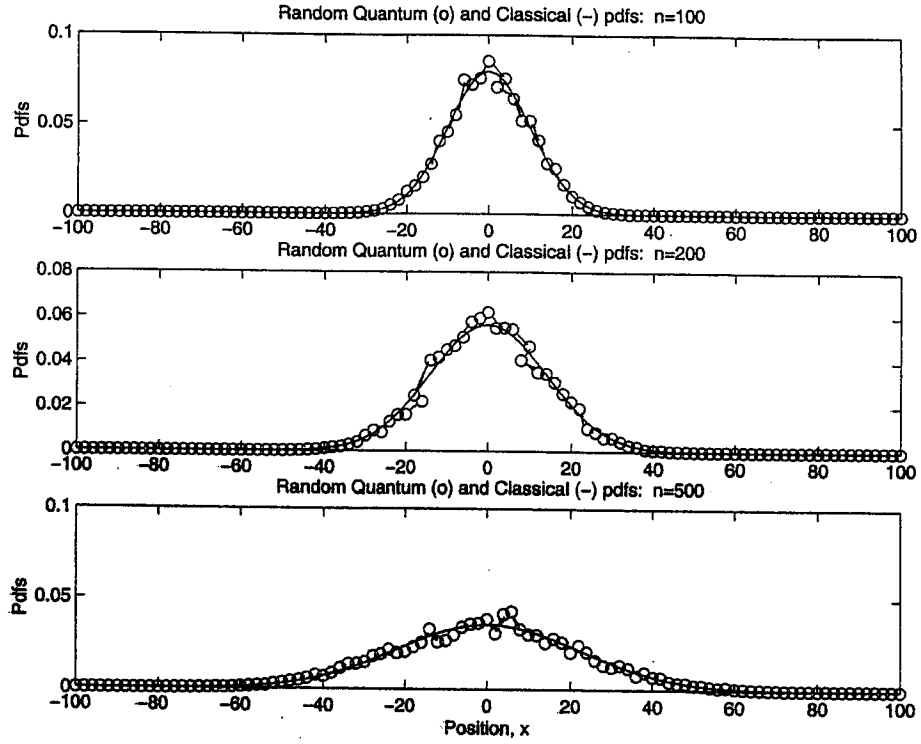

Figure 12: Classical versus Random Quantum pdfs

It does appear that the 'random' quantum walk pdf spreads out as the final time iteration is increased, similarly to the classical random walk. Does this suggest the occurrence of a form of quantum Brownian Motion? Let us investigate this phenomenon further with the expectation values and variance.

4.2.2 Expectation

In the classical random walk it is possible to graph the exact position of the point at any time. In this way we observe a realisation of Brownian Motion. Such realisations are unobservable, however, with the quantum analogy, since we are unable to determine a precise position for any point at any given time in the quantum regime. It is possible, though, to

analyse instead, the quantum expectation value of the point after each iteration and compare this to the classical results.

The discrete quantum expectation value was calculated as follows:

$$\mathbb{E}(K_n = k) = \sum_{k=-n}^{n} kp_k$$

where K_n is the discrete position random variable, p_k is the probability of the point being in position k, after n iterations.

Discrete expectation values can then be plotted against the iteration number in order to give a graph of the changing expectation value of the point's position. See figure 13 for the expectation values of a 'random' quantum walk with initial position vector, $\psi^{(0)} = (1\ 0)^T$.

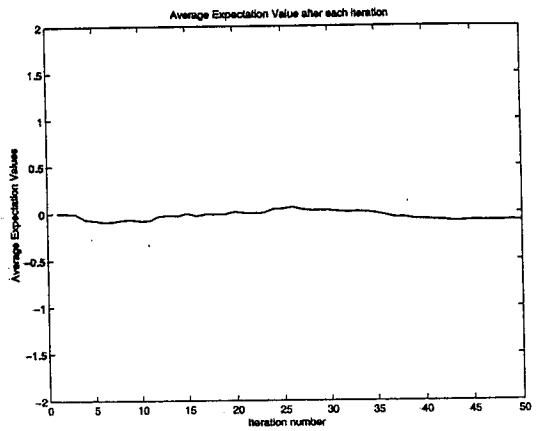

Figure 13: 'Random' Quantum Expectation Values

Classically we expect that the expectation value of position will be $\mathbb{E}(X_n) = 0$. In the 'random' quantum walk expectation graph we observe that the expected position of the particle does oscillates about zero. It can be assumed that as the number of repetitions is increased for greater accuracy, the discrepancy will become infinitesimally small, such that $\mathbb{E}(K_n) \to 0$. In the 'Pure' quantum walk, with initial state $\psi^{(0)} = (1\ 0)^T$, the expectation value appeared to be linearly increasing with time. Hence the 'random' quantum walk expectation values appear to mirror the classical rather than the 'pure' quantum results more closely.

4.2.3 Variance

Let us also investigate the mean square deviation of the position from the mean. The quantum variance was calculated as follows:

$$Var(K_n = k) = \mathbb{E}(K_n)^2 - (\mathbb{E}K_n)^2$$

In the classical regime, the expectation value $\mathbb{E}(X_n)$ is equal to zero. Therefore classically, $Var(X_n) = \mathbb{E}(X_n^2) = n$, assuming that each iteration step is one unit.

Averaging over a sufficiently large number of iterations, the 'random' quantum variance closely resembles the linearly increasing classical variance graph (minus the relatively small $(\mathbb{E}K_n)^2$ values).

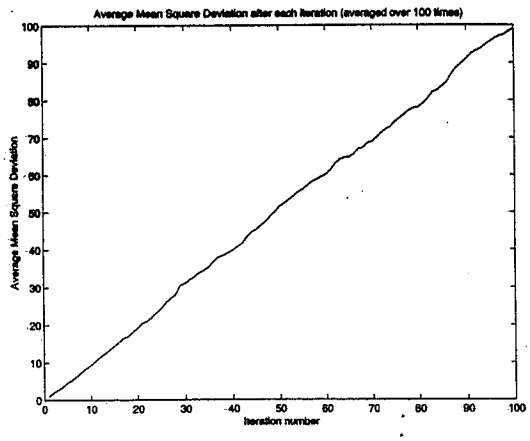

Figure 14: 'Random' Quantum Variance with $\psi^{(0)} = (1\ 0)^T$

It does appear that the probability distribution, expectation and variance of the 'random' quantum walk resemble the classical results more closely than those of the 'pure' quantum walk. In order to support these numerical findings, a more analytic appreciation of the 'random' quantum system at any time can be determined by examination of the density matrix.

4.3 Probability Density Matrix

The quantum density matrix can perhaps be considered as the quantum analogue of the classical transition matrix; where

$$P_n(i,j) = \mathbb{P}(X_n = j | X_0 = i)$$

One property of the classical transition matrix is that the eigenvector corresponding to $\pi = \pi P_n$ is a stationary or limiting distribution. So that π indicates the steady state of the system, independent of the initial state.

In the 'pure' quantum case, 'unitarity' prevents the walk from reaching a steady state. However, a limiting distribution that depends on the initial state can be found, for a continuous quantum walk, as given by Childs et al. [4]. This involves expansion over the energy eigenstates of the system to form an analytic probability distribution for the 'pure' quantum walk.

Although the limiting distribution is not so easily calculated for the 'random' quantum walk, the quantum density operator can still divulge useful information. The eigenvalues of the quantum density matrix can be interpreted as the probability of the system being in its corresponding eigenstate. This is an indication of the probable form of the wave function at any time, n.

Quantum mechanically, the density operator is defined as a unitary matrix;

$$\rho^{(n)} = |\psi^{(n)}\rangle\langle\psi^{(n)}|$$

Given an initial position vector $\psi^{(0)}$, then, it is possible to construct the density matrix for any iteration. As previously shown, each iteration of the quantum walk is performed by the unitary operator V. Such that,

$$\psi^{(n)} = V\psi^{(n-1)}$$

$$\text{correspondingly} \quad \psi^{(n)\dagger} = \psi^{(n-1)\dagger} V^{-1}$$

$$\text{Now since} \quad \rho^{(n)} = \psi^{(n)}\psi^{(n)\dagger} \quad \Rightarrow \rho^{(n)} = V\psi^{(n-1)}\psi^{(n-1)\dagger}V^{-1} \quad = V\rho^{(n-1)}V^{-1}$$

In the 'Random' Quantum case, the walk progression operator includes a classically random element, $r_n = \pm 1$, with equal probability. We must consider the combined effect of the two unitary operators, $V_\pm$, each with fifty percent probability of being applied at each iteration, independently. Therefore the density matrix for the 'Random' quantum walk becomes,

$$\rho^{(n)} = \frac{1}{2}V_+\rho^{(n-1)}V_+^{-1} + \frac{1}{2}V_-\rho^{(n-1)}V_-^{-1}$$

The initial density matrix ρ_0 is calculated from the assigned initial state.

$$\rho^{(0)} = |\psi^{(0)}\rangle\langle\psi^{(0)}| \quad = \psi^{(0)}\psi^{(0)\dagger}$$

Thus if the initial spin state was chosen as $\psi^{(0)} = (\ u\ v\ )^T$, the density matrices, its eigenvectors and eigenvalues can be determined, as found in Appendix 3 for n=0,1,2,3.

Assigning $|u|^2 + |v|^2 = 1$ and $\alpha = \bar{u}v - \bar{v}u$, it is determined that the general form of the eigenvalues, λ_i is;

$$\lambda_i = \frac{1}{b_i} \pm \frac{1}{b_i}|\alpha|$$

where b_i are constants, dependent on the initial conditions and the corresponding eigenstates. The eigenstates for all $\alpha \neq 0$, consist of two non-zero components; 1 and $\pm i$. And for all $\alpha = 0$, that is u and v are real, the eigenstates' only non-zero component is 1. The placement of these components within the vectors and the size of their corresponding eigenvalues, is an indication of the probable position of the quantum point.

For example if the initial spin state was chosen as in the positive Z dimension;

$$\psi^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad => \quad \rho^{(0)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues of $\rho^{(n)}$ for n=0,1,2,3 are as follows:

n=0							1							
n=1					$\frac{1}{2}$	0	0	$\frac{1}{2}$						
n=2			$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$				
n=3	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{8}$	0	0	$\frac{1}{8}$		

Figure 15: Eigenvalues of density matrix, $\alpha = 0$

This formation, except for n=0, is the same for all real valued u and v. It is clear that this eigenvalue distribution will approach that of the the classical discrete probability distribution, as n becomes large. The density eigenvalues indicate the production of a bell-shaped continuous probability curve, somewhat similar to the classical random walk pdf. However, the progression of this curve in time is not immediately clear. It is therefore beneficial to investigate the entropy of the system, as will be further explained.

4.4 Entropy- Disinformation

Another interesting value which can be determined by investigation of the density matrix, is the Entropy, H, of the system. That is, the loss of information the system accrues after n iterations.

$$H = -\text{trace}(\rho \log(\rho))$$

$$=> H = -\sum_i \lambda_i \log(\lambda_i)$$

where λ_i are the eigenvalues of ρ.

The entropy versus the number of iterations is as follows.

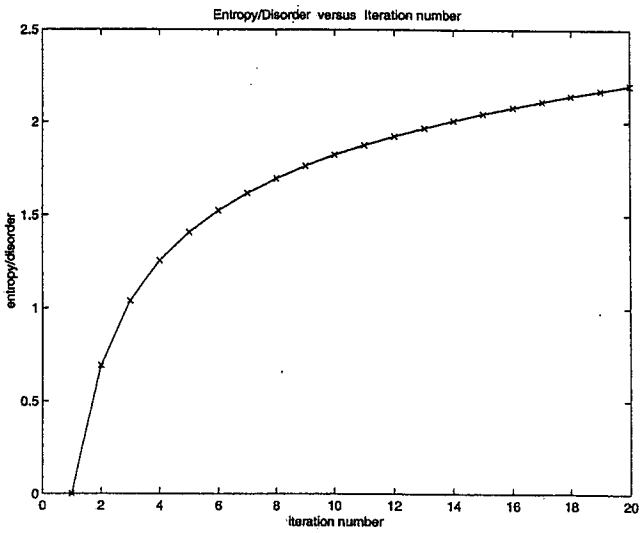

Figure 16: Entropy versus time

The graph above displays only 20 iterations, yet this is sufficient to see a definite relation between the entropy and time, of the system at hand.

It appears that as time progresses, the entropy or the loss of information is increasing. This corresponds to the linearly increasing variance graph and the change of the quantum pdfs with time, as produced earlier. As the number of iterations is increased, the pdf graph appears to spread out. This increase in 'spread' indicates a growing uncertainty of "where" the point actually is. That is, as the iterations increase so does the loss of information, entropy.

Therefore, to say that the disorder/entropy is rising, suggests that as the number of steps in the quantum random walk increases, the 'random' quantum pdf should exhibit linearly increasing variance. Such evidence supports the behaviour of the 'random' quantum walk observed numerically.

4.5 Is Quantum Brownian Motion found?

The numerically observed pdfs, expectation and variance graphs of the 'random' quantum walk indicate the development of a bell-shaped curve that spreads out over time, closely

resembling the classical Gaussian. The analytic examination of the density matrix and entropy, supports these results; the 'random' quantum walk appears to behave similarly to the classical random walk, rather than the 'pure' quantum walk.

All results combined, there does appear to be considerable evidence that the introduction of a random element into the 'pure' quantum walk on a line produces classical-like results. While this suggests the existence of quantum Brownian Motion for this particular case of rotation operator, further numerical and analytical investigation is necessary.

Chapter 5

General Hybrid Walks

In the construction of a 'random' quantum walk, the classically random element is insinuated into the rotation operator. The choice of rotation operator, is arbitrary, provided it is a unitary (2X2) matrix. The rotation matrix of the previous chapter was classically analogous, as it had two simple opposing rotation matrices, however, similar results should arise from other suitable rotation operators.

If a slightly differing operator is inserted into the system, do we observe the same results? How far from the purely quantum case must we go in order to see the effects of the classical randomness in the system?

5.1 Asymmetrical Hybrid Walk

The most practical initial approach to this question is to examine hybrid operators which can vary the randomness between the purely quantum and random quantum case already observed.

With the 'random' quantum case just described, the operators utilised were;

$$R_n = \frac{1}{\sqrt{2}}(I + ir_n\sigma_2)$$

$$R_n = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & r_n \\ -r_n & 1 \end{pmatrix}$$

where $r_n = +1$ or $r_n = -1$ is randomly assigned with equal probability at each iteration.

The 'pure' quantum case can be similarly represented and constructed by setting

$$R_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -r_n \\ r_n & 1 \end{pmatrix}$$

where $r_n = +1$ or $r_n = +1$ for every iteration.
That is, the 'pure' quantum case is represented in the 'random' quantum regime by allowing r_n to take the value of $+1$ only. Thus the two cases can be formed through the same computational processes.

Let us symbolise the probable rotation operators as;

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad <==> \quad \uparrow$$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad <==> \quad \downarrow$$

Thus we can conceptualise;

$$\text{Pure Quantum rotation operator} \quad R_n = \begin{cases} \uparrow & r_n = +1 \\ \uparrow & r_n = -1 \end{cases}$$

$$\text{Random Quantum rotation operator} \quad R_n = \begin{cases} \uparrow & r_n = +1 \\ \downarrow & r_n = -1 \end{cases}$$

It is possible, then, to construct a hybrid operator that can gauge the results as we move from the 'pure' quantum to the 'random' quantum case.

In order to produce a system which can represent a hybrid process between these two, it is necessary to construct an asymmetric rotation operator. That is, the first possible rotation matrix is fixed as $\uparrow$ and the second possible rotation matrix is variable between $\uparrow$ and $\downarrow$.

5.1.1 Introduction of a new variable ϵ

Let us introduce a new parameter ϵ, which is assigned a value that is fixed for all iterations. This parameter determines the form of the second possible rotation matrix, such that;

$$R_n = \begin{cases} \frac{1}{\sqrt{2}}(I + i\sigma_2) & r_n = +1 \\ \frac{1}{\sqrt{1-(\epsilon-1)^2}}(I + i(1-\epsilon)\sigma_2) & r_n = -1 \end{cases}$$

$$R_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & r_n = +1 \\ \frac{1}{\sqrt{1-(\epsilon-1)^2}} \begin{pmatrix} 1 & -1+\epsilon \\ 1+-\epsilon & 1 \end{pmatrix} & r_n = -1 \end{cases}$$

Such that if $\epsilon = 0$ the system resembles the 'pure' quantum case and if $\epsilon = 2$ we find the 'random' quantum case. For values of ϵ, $0 < \epsilon < 2$ we can examine the hybrid cases which occur between these two.

5.1.2 Asymmetric Hybrid pdf

Let us examine the probability density functions of this asymmetric hybrid walk, as the degree of randomness is increased.

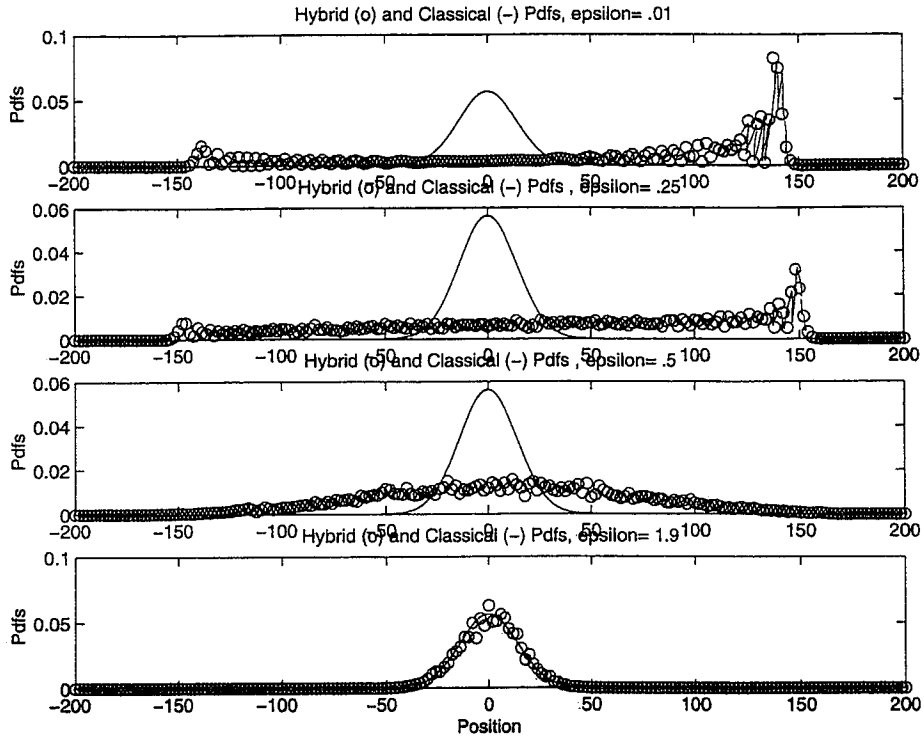

Figure 17: Asymmetric Hybrid pdf, n=200

We see a gradual change from a two-peaked quantum pdf to the Gaussian-like shape of the 'random' quantum pdf, as the degree-of-randomness ϵ is increased from zero to two. More investigation is necessary, perhaps for a more symmetrical application of the degree-of-randomness parameter, ϵ.

5.2 Symmetric Hybrid Walks

In order to investigate this apparent 'classicalisation' of the quantum system from any amount of randomisation, we should perhaps form a more symmetrical alternating operator. For example,

$$R_n = (I + i\sigma_2)(I + i\epsilon r_n \sigma_k)\frac{1}{\sqrt{2}\sqrt{1+\epsilon^2}}$$

where if k=2 we have commuting factors and if k=1 we have non-commuting factors.

For the commuting case, we can write

$$R_n = \begin{cases} \frac{1}{\sqrt{2}\sqrt{1+\epsilon^2}} \begin{pmatrix} 1-\epsilon & 1+\epsilon \\ -1-\epsilon & 1-\epsilon \end{pmatrix} & r_n = +1 \\ \frac{1}{\sqrt{2}\sqrt{1+\epsilon^2}} \begin{pmatrix} 1+\epsilon & 1-\epsilon \\ -1+\epsilon & 1+\epsilon \end{pmatrix} & r_n = -1 \end{cases}$$

For the non-commuting case, we can write

$$R_n = \begin{cases} \frac{1}{\sqrt{2}\sqrt{1+\epsilon^2}} \begin{pmatrix} 1+i\epsilon & 1+i\epsilon \\ -1+i\epsilon & 1-i\epsilon \end{pmatrix} & r_n = +1 \\ \frac{1}{\sqrt{2}\sqrt{1+\epsilon^2}} \begin{pmatrix} 1-i\epsilon & 1-i\epsilon \\ -1-i\epsilon & 1+i\epsilon \end{pmatrix} & r_n = -1 \end{cases}$$

Unlike the previous hybrid, these operators cannot be clearly related to the 'random' quantum walk rotation operators.

5.2.1 Symmetric Hybrid pdfs

Let us examine the pdfs obtained from application of these symmetric hybrid operators, for different values of ϵ.

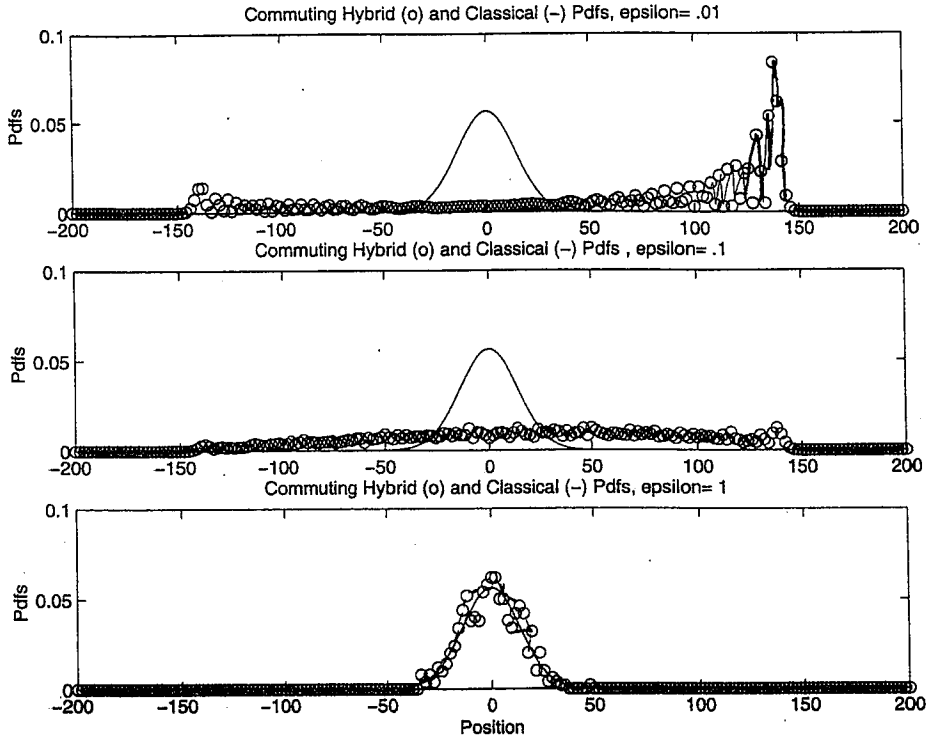

Figure 18: Non-Commuting Symmetric Hybrid Pdfs, n=200

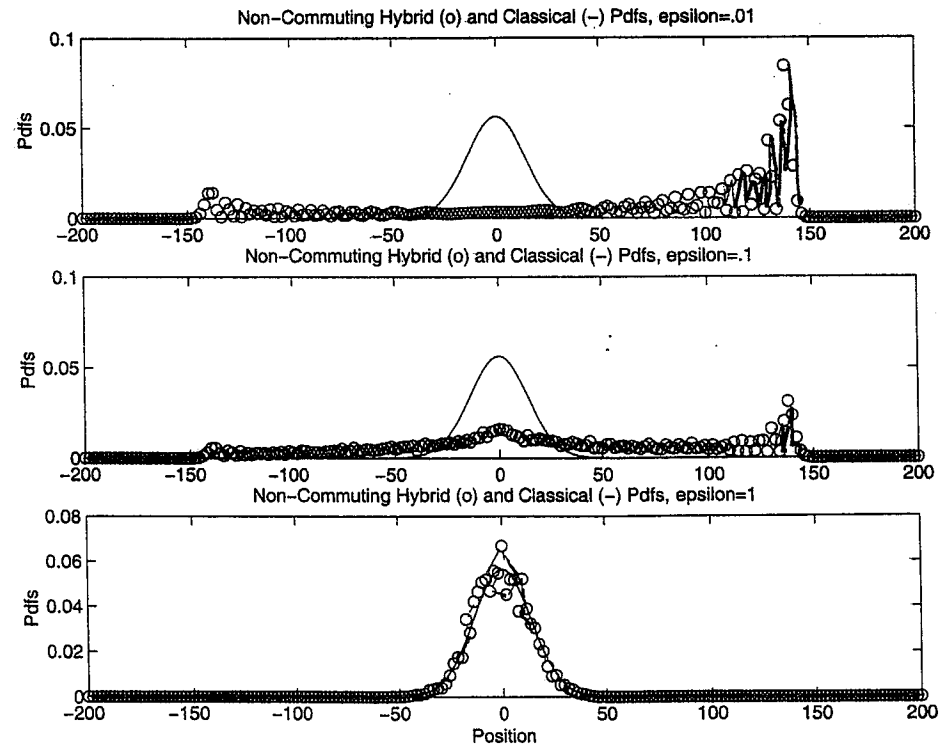

Figure 19: Commuting Symmetric Hybrid Pdfs, n=200

The gradual change from the quantum two-peaked pdf at $\epsilon = 0$ to a Gaussian-like shape as $\epsilon > 1$ appears to occur for the symmetric hybrid walks also.

First to note is that for values of $\epsilon > 1$, the pdf observed appears to form centrally peaked graph. This may indicate that any symmetric operator, with a random parameter dominant enough, gives a bell-shaped pdf. In order to determine if this formation is similar to the classical pdf, it is necessary to discover if the bell-like shape tends to spread out as the iterations are increased.

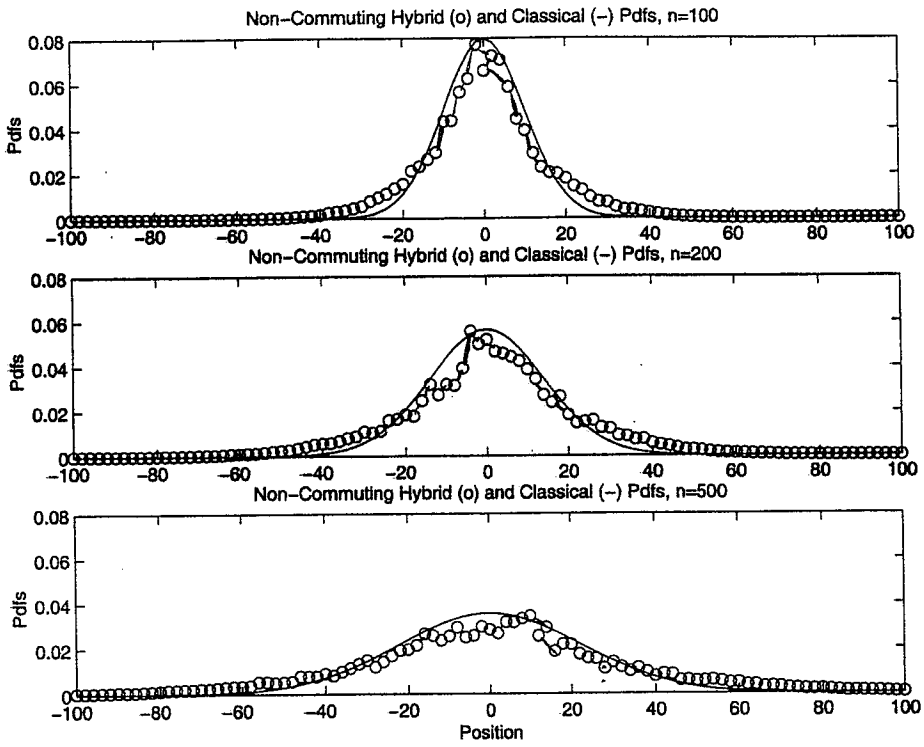

Figure 20: Non-Commuting Symmetric Hybrid Pdfs, $\epsilon = 2$

It does appear that for $\epsilon = 2$ (in the non-commuting symmetric hybrid walk), as n is increased, the bell-shaped pdf does spread out as expected. Hence, the hybrid walk pdf may approximate a Gaussian. This supports the hypothesis that any rotation operator with a sufficient degree of randomness will exhibit classical random walk behaviour. The numerical results given here, however, are insufficient for any conclusions to be made. More numerical

and analytic examination, perhaps of a general density matrix, is necessary.

Secondly, the results produced above lead to the question; for what values of ϵ is this classical-like behaviour observed? What degree of randomness is necessary in order for the quantum to tend towards the classical? The answer to this question is gained through observation of the process over increasing time iterations.

Let us examine the transformation in time of the hybrid walk pdf, with $\epsilon = .1$.

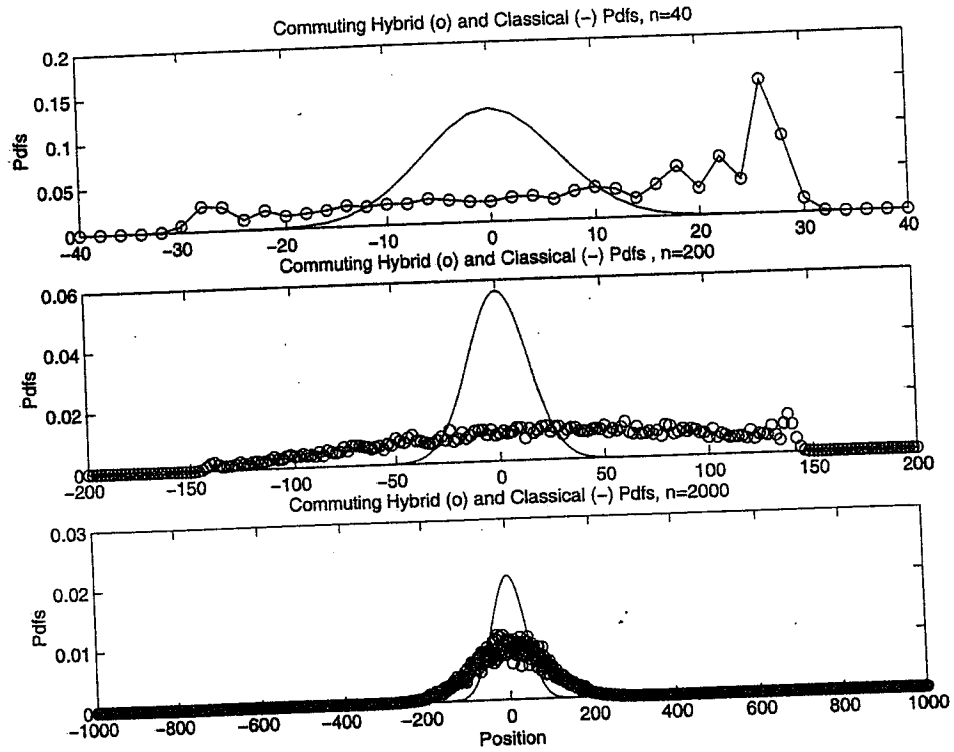

Figure 21: Commuting Symmetric Hybrid Pdfs, $\epsilon = .1$

For smaller values of n, the pdf of the hybrid walk with $\epsilon = .1$ resembles a sligthly altered two-peaked quantum pdf. As the iteration number is increased, however, the peaks appear to merge into a centrally positioned single peak, and eventually form a Gaussian-like pdf that continues to 'spread-out'.

Such results as these suggest that an operator with any degree of randomness, regardless how insignificant seeming, will produced classical-like probability behaviour for a sufficiently large number of iterations.

These results appears to imply a relationship between the degree-of-randomness, ϵ, and the iteration time, n. Examination of this relationship can be achieved through observation of the expectation and variance of the hybrid walk in time, for different values of ϵ. Through this, it may be possible to numerically determine an approximate transition period, from the quantum to the classical-like results.

5.2.2 Relationship between 'Degree-of-Randomness' and Time

In order to gauge the combined effects of n and ϵ on the form of the pdf, examination of the expectation values and variance were made.

For the 'pure' quantum walk previously investigated ($\psi^{(0)} = (1\ 0)^T$) we observed a steadily increasing expectation value, where as the classical and 'random' quantum walks revealed a expectation value of approximately zero. Thus the resulting hybrid expectation graph will gradually change from a linearly increasing curve to flatten out. The period in which this occurs will correspond to the 'symmetrising' of the pdf, and thus the beginning of the transition from the quantum to the classical-like results.

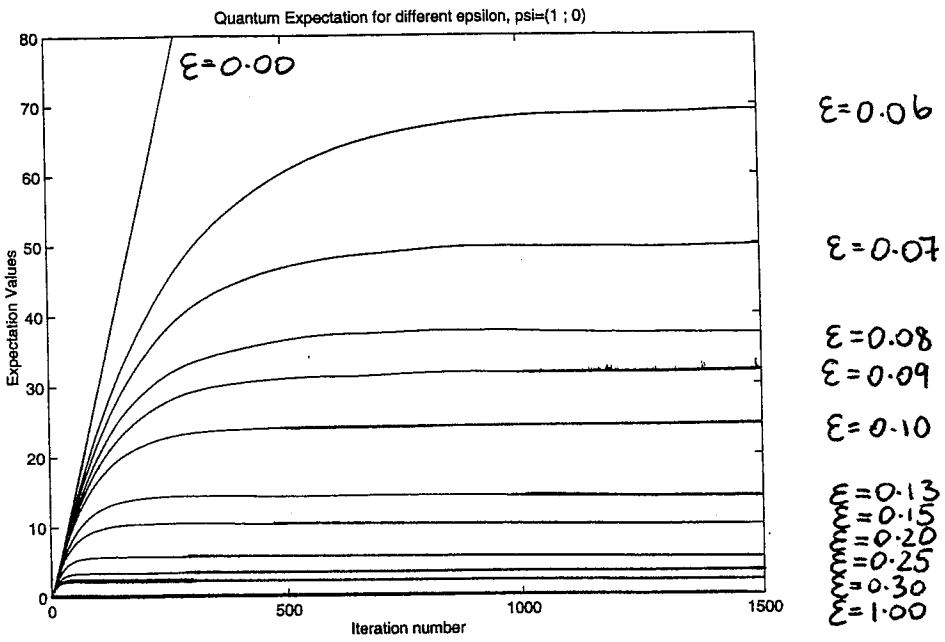

Figure 22: Non-Commuting Symmetric Hybrid Expectation Value

Similarly, the variance can be observed as the change from a (quantum) parabola to a (classical) linearly increasing line.

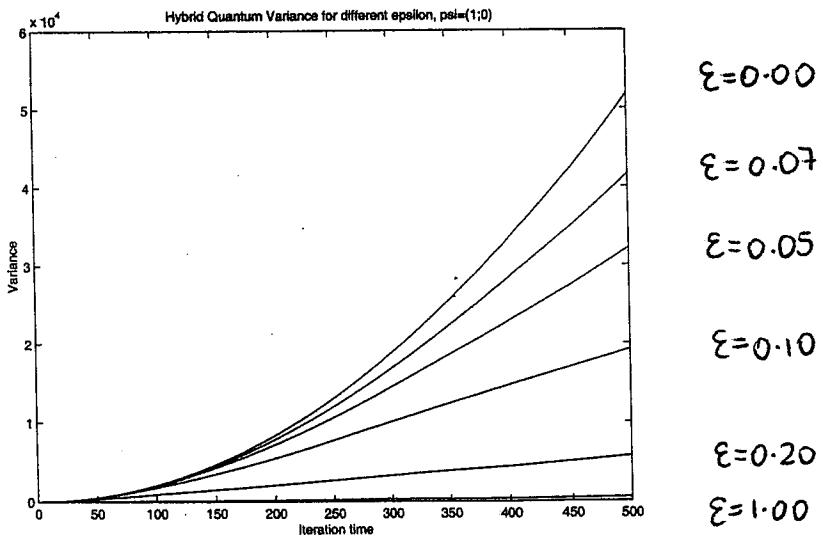

Figure 23: Non-Commuting Symmetric Hybrid Variance

An estimate of the 'cross-over' period can be approximately calculated through numerical investigation of graphs such as these. The apparent transition from quantum to classical

is not instantaneous, but will occur gradually. However, as n becomes large, the resultant graphs will indicate an approximate initial 'cross-over' time, in relation to the other values of ϵ. Following is a graph that relates the degree-of-randomness, ϵ, to the approximate initial transition time.

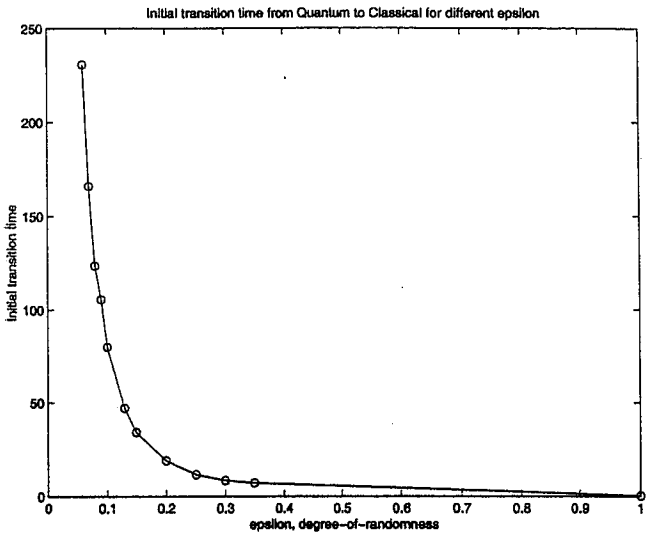

Figure 24: Transition time from Quantum to Classical, for different ϵ

The graphs appear to demonstrate a decreasing non-linear curve. As the degree-of-randomness is increased, the time before the quantum results begin to exhibit classical-like behaviour decreases. This relationship is intuitive: A hybrid quantum system will tend to exhibit classical results more readily, as the classical element becomes more influential in the process.

The numerical evidence observed here appears to support the hypothesis that the addition of any degree of classical randomness into a quantum random walk, will produce classical-like results, for sufficiently large time iterations.

Chapter 7

Conclusion

Numerical evidence presented in this thesis appears to support the hypothesis that a quantum walk with any degree of classical randomness adopted into the rotation operator, will exhibit classical random walk behaviour, over a sufficiently large time. The numerical evidence given here, however, is insufficient for any comprehensive conclusions to be made. This field of study has many exciting applications, thus an extensive analytic and practical investigation of the hybrid random walk would be enlightening and advantageous.

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