Vector Operators and a Polynomial Identity for $SO(n)$

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It is shown that if α denotes an $n \times n$ antisymmetric matrix of operators $\alpha_{pq}, p, q = 1, 2, \ldots, n$, which satisfy the commutation relations characteristic of the Lie algebra of $SO(n)$, then α satisfies an nth degree polynomial identity. A method is presented for determining the form of this polynomial for any value of n. An indication is given of the simple significance of this identity with regard to the problem of resolving an arbitrary n-vector operator into n components, each of which is a vector shift operator for the invariants of the $SO(n)$ Lie algebra.

1. INTRODUCTION

The structure of 3-vector operators in quantum theory was investigated by Dirac,[1] Güttinger, and Pauli,[2] who considered the matrix elements of such operators in an angular momentum basis. Later Wigner[3] indicated the possibility of a systematic treatment of any set of operators transforming according to an irreducible representation of the rotation group, and the calculus of tensor operators was subsequently developed by Racah.[4] Some of the results of these investigations, and their application to calculations in the quantum theory of atomic spectra, can be found in the books by Condon and Shortley[5] and Slater.[6]

Racah[7] and Biedenharn[8] have emphasized the desirability of finding, in the case of other semisimple groups, the generalization of these and other results in the theory of angular momentum, or $SU(2)$. We present here some results in the theory of $SO(n)$, or, more accurately, of its universal covering group [for convenience, this group is subsequently referred to as $SO(n)$], relating in particular to the description of n-vector operators. Even in the much studied case $n = 3$, our approach has, we believe, some novel and attractive features.

We are concerned with the general situation where one is given a set of operators θ_p, $\alpha_{qr}\ (= -\alpha_{rq})$, $p, q, r = 1, 2, \ldots, n$, satisfying the commutation relations

$$[\alpha_{pq}, \alpha_{rs}] = \delta_{qr}\alpha_{ps} + \delta_{ps}\alpha_{qr} - \delta_{pr}\alpha_{qs} - \delta_{qs}\alpha_{pr}, \tag{1}$$

$$[\theta_p, \alpha_{qr}] = \delta_{pq}\theta_r - \delta_{pr}\theta_q. \tag{2}$$

In particular, the α_{pq} could be anti-Hermitian operators acting in a Hilbert space, in which case they form the generators of a unitary representation, in general reducible, of $SO(n)$. Then $\boldsymbol{\theta}$ $[= (\theta_1, \theta_2, \ldots, \theta_n)]$ is an n-vector operator acting within the corresponding representation space.

In such a case, the Casimir operator $\sigma_2 = \alpha_{pq}\alpha_{qp}$ can be expressed in the form[9]

$$\sigma_2 = 2\Lambda_1(\Lambda_1 + n - 2) + 2\Lambda_2(\Lambda_2 + n - 4) + \cdots + 2\Lambda_m(\Lambda_m + n - 2m), \tag{3}$$

where m is the integral part of $\frac{1}{2}n$ and the eigenvalues λ_i of the operators Λ_i, which serve to label the irreducible components of the representation of $SO(n)$, are either all integers or all half-odd integers and satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0, \quad n = 2m + 1,$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m| \geq 0, \quad n = 2m. \tag{4}$$

When $n = 3$, an established result[1,2] is that $\boldsymbol{\theta}$ can be resolved into three components, each of which is a 3-vector shift operator for Λ_1, the magnitude of the angular momentum. Thus

$$\boldsymbol{\theta} = \boldsymbol{\theta}^+ + \boldsymbol{\theta}^0 + \boldsymbol{\theta}^-,$$

where

$$\Lambda_1\boldsymbol{\theta}^0 = \boldsymbol{\theta}^0\Lambda_1, \quad \Lambda_1\boldsymbol{\theta}^\pm = \boldsymbol{\theta}^\pm(\Lambda_1 \pm 1).$$

The results obtained by Bhabha,[10] in investigations of 4-vector operators within finite-dimensional representations of $SO(3, 1)$, enable us to deduce that, for $n = 4$,

$$\boldsymbol{\theta} = \boldsymbol{\theta}_1^+ + \boldsymbol{\theta}_1^- + \boldsymbol{\theta}_2^+ + \boldsymbol{\theta}_2^-,$$

where

$$\Lambda_1\boldsymbol{\theta}_1^\pm = \boldsymbol{\theta}_1^\pm(\Lambda_1 \pm 1), \quad \Lambda_2\boldsymbol{\theta}_1^\pm = \boldsymbol{\theta}_1^\pm\Lambda_2, \quad \Lambda_1\boldsymbol{\theta}_2^\pm = \boldsymbol{\theta}_2^\pm\Lambda_1,$$

$$\Lambda_2\boldsymbol{\theta}_2^\pm = \boldsymbol{\theta}_2^\pm(\Lambda_2 \pm 1).$$

It is not difficult (see Appendix B) to deduce the generalization of these results for $n = 3$ and 4. Thus when $n = 2m + 1$, θ can be resolved into components $\boldsymbol{\theta}^0, \boldsymbol{\theta}_i^+, \boldsymbol{\theta}_i^-$, $i = 1, 2, \ldots, m$, where

$$\Lambda_i\boldsymbol{\theta}^0 = \boldsymbol{\theta}^0\Lambda_i, \quad \Lambda_i\boldsymbol{\theta}_j^\pm = \boldsymbol{\theta}_j^\pm(\Lambda_i \pm \delta_{ij}), \tag{5}$$

while, in the case $n = 2m$, the result is the same, except that θ^0 does not occur.

In what follows, it is convenient to think of α_{pq} as the element in the pth row and qth column of an antisymmetric $n \times n$ matrix of operators. We denote this matrix by α and by α^k and $\overline{\alpha^k}$ the matrices whose pqth elements are

$$(\alpha^k)_{pq} = (\alpha^{k-1})_{pr}\alpha_{rq}, \quad k = 2, 3, \ldots,$$

$$(\overline{\alpha^k})_{pq} = (\alpha^k)_{qp}.$$

Furthermore (cf. σ_2), we define σ_k by

$$\sigma_k = \text{tr}\,[\alpha^k] = (\alpha^k)_{pp}.$$

We shall show that, as a consequence of the relations (1), α satisfies an nth-degree polynomial identity, of the form

$$F_n(\alpha) = \alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0, \quad (6)$$

where the coefficients a_k, $k = 1, 2, \ldots, n$, are invariants of the $SO(n)$ Lie algebra.

The existence of this identity is in no way dependent on the existence or nonexistence, within the representation space for α, of an n-vector operator $\boldsymbol{\theta}$. However, when such a θ does exist, its resolution into n n-vector shift operators for the $SO(n)$ invariants, as in Eqs. (5) above, can most readily be achieved with the use of this identity, as we shall see.

We are mainly concerned with the determination of the form of the coefficients a_k in Eq. (6), as functions of $\sigma_2, \sigma_3, \ldots$, which are in turn functions, also to be determined, of $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$. Before proceeding, however, we make the following general remarks concerning the identity expressed in Eq. (6).

It is clearly an analog of the Cayley–Hamilton identity for an $n \times n$ matrix of complex numbers. There are, however, some interesting differences.

In particular, suppose we define the determinant of an $n \times n$ matrix A of noncommuting elements, by

$$\det(A) = (1/n!)\epsilon_{ij\ldots m}\epsilon_{pq\ldots t}A_{ip}A_{jq}\cdots A_{mt},$$

where $\epsilon_{ij\ldots m}$ is the alternating tensor, with $\epsilon_{12\ldots n} = 1$. Then in the present context, we find

$$\det(\alpha - \rho I) = \rho^n + a'_1\rho^{n-1} + \cdots + a'_n, \quad (7)$$

where ρ is an arbitrary complex number, I is the $n \times n$ unit matrix, and a'_k, $k = 1, 2, \ldots, n$, like a_k, is an invariant of $SO(n)$. However, we find that, in general, $a'_k \neq a_k$, in contrast with the case when α is a matrix of complex numbers.

The existence of the polynomial (7) is comparatively well known, having been discussed in studies of Lie algebras by Killing[11] before the turn of the century and, more recently, by Racah[7] and Biedenharn.[8] Much less, it seems, is known of identities of the form in Eq. (6).

This equation actually expresses n^2 identities in the elements of α. Lehrer-Ilamed[12] has shown that n^2 identities of more general form are satisfied by the elements of any $n \times n$ matrix, provided these elements belong to a free associative algebra. Here we are dealing with a special case, where the algebra is in fact a Lie algebra, whose structure constants are such that these identities can be expressed in the simple form of a polynomial identity in α. [One of the authors (H.S.G.) has now determined similar identities for $Sp(n)$ and $SU(n)$.]

In the course of investigating certain identities satisfied by elements of any representation of the Lie algebra of $SU(3)$, Lehrer-Ilamed[13] has utilized similar generalized Cayley–Hamilton identities satisfied by some elements of the algebra. (See also Racah.[14]) However, to our knowledge, the identity (6) for $SO(n)$ has not previously been presented, even for the case $n = 3$.

2. VECTOR SHIFT OPERATORS AND THE POLYNOMIAL IDENTITY

Considering a system of operators $\boldsymbol{\theta}, \alpha$ as in Eqs. (1)–(5), we find that θ^0, θ_i^+, and θ_i^- are eigenvectors of the generator matrix α, in the sense that if θ_τ represents any one of these n operators,

$$\alpha\boldsymbol{\theta}_\tau = d_\tau\boldsymbol{\theta}_\tau,$$

i.e.,

$$\alpha_{pq}\theta_{\tau q} = d_\tau\theta_{\tau p},$$

where d_τ is an invariant of $SO(n)$. This follows from the fact that, for any vector operator $\boldsymbol{\theta}_\tau$,

$$[\sigma_2, \boldsymbol{\theta}_\tau] = 2(2\alpha - n + 1)\boldsymbol{\theta}_\tau.$$

For, in view of Eqs. (3) and (5) above, σ_2 commutes with θ^0, so that

$$(\alpha - \tfrac{1}{2}n + \tfrac{1}{2})\boldsymbol{\theta}^0 = 0. \quad (8)$$

Also,

$$[\sigma_2, \boldsymbol{\theta}_i^+] = 2[\Lambda_i(\Lambda_i + n - 2i), \boldsymbol{\theta}_i^+]$$

$$= 2(2\Lambda_i + n - 2i + 1)\boldsymbol{\theta}_i^+,$$

so that

$$(\alpha - \tfrac{1}{2}n + 1)\boldsymbol{\theta}_i^+ = (\Lambda_i + \tfrac{1}{2}n - i)\boldsymbol{\theta}_i^+. \quad (9)$$

Similarly,

$$(\alpha - \tfrac{1}{2}n + 1)\boldsymbol{\theta}_i^- = -(\Lambda_i + \tfrac{1}{2}n - i)\boldsymbol{\theta}_i^-. \quad (10)$$

In Appendix A we show that α satisfies a polynomial identity of the general form of Eq. (6). The results (8)–(10) then allow us to write the identity in more precise form, effectively determining a_k, $k = 1, 2, \ldots, n$ as a function of Λ_i, $i = 1, 2, \ldots, m$. We must have

$$F_n(\alpha) = 0, \quad (11)$$

where

$$F_n(\alpha) = G_m(a^2), \qquad n = 2m, \quad (12)$$

$$F_n(\alpha) = (a - \tfrac{1}{2})G_m(a^2), \qquad n = 2m + 1, \quad (13)$$

and

$$a = \alpha - \tfrac{1}{2}n + 1,$$

$$G_m(a^2) = \prod_{i=1}^{m} [a^2 - (\Lambda_i + \tfrac{1}{2}n - i)^2]. \quad (14)$$

Conversely, once the results (11)–(14) are known, one can see why and how an arbitrary n-vector operator θ can be resolved in the manner indicated by Eqs. (5). Thus, in the case $n = 2m + 1$, we use Eq. (11) in the obvious way to define projection operators $P^0(\alpha), P_i^+(\alpha), P_i^-(\alpha)$, $i = 1, 2, \ldots, m$, which are polynomials of the $(n-1)$th degree in α and which satisfy

$$(\alpha - \tfrac{1}{2}n + \tfrac{1}{2})P^0 = 0,$$

$$[\alpha - \tfrac{1}{2}n + 1 \mp (\Lambda_i + \tfrac{1}{2}n - i)]P_i^{\pm} = 0,$$

$$P^0 P_i^{\pm} = P_i^{\pm} P^0 = P_i^+ P_j^- = P_i^- P_j^+ = 0,$$

$$(P^0)^2 = P^0, \quad P_i^{\pm} P_j^{\pm} = \delta_{ij} P_i^{\pm},$$

$$P^0 + \sum_{i=1}^{m} (P_i^+ + P_i^-) = 1. \tag{15}$$

Then the required resolution is

$$\boldsymbol{\theta} = \boldsymbol{\theta}^0 + \sum_{i=1}^{m} (\theta_i^+ + \theta_i^-),$$

with

$$\theta^0 = P^0\theta, \quad \theta_i^{\pm} = P_i^{\pm}\boldsymbol{\theta}. \tag{16}$$

The case $n = 2m$ is similar, except that P^0 and $\boldsymbol{\theta}^0$ do not occur.

We do not go into details here, but mention that in order to confirm that Eqs. (5) follow from Eqs. (15) and (6), it is not sufficient to consider the commutators of $\boldsymbol{\theta}^0, \boldsymbol{\theta}_i^{\pm}$ only with σ_2. Rather one needs to calculate the commutators of these vectors with a complete set of invariants, which, like σ_2 but unlike Λ_i, are *explicitly* constructed from the set of α_{pq}. Finally, one must know the expression for each member of this complete set in terms of the Λ_i. We return to this last point in Sec. 5.

3. SYMMETRIC AND ANTISYMMETRIC POLYNOMIALS IN α

If the matrix polynomial $f(\alpha)$ is symmetric, i.e., if $\overline{f(\alpha)} = f(\alpha)$, then

$$g(\alpha) = (\alpha - \tfrac{1}{2}n)f(\alpha) + \tfrac{1}{2}\,\mathrm{tr}\,[f(\alpha)] \tag{17}$$

is antisymmetric, i.e., $\overline{g(\alpha)} = -g(\alpha)$.

Furthermore, if $g(\alpha)$ is antisymmetric, then

$$h(\alpha) = (\alpha - \tfrac{1}{2}n + 1)g(\alpha) \tag{18}$$

is symmetric.

The proof is as follows. Since $f_{pq}(\alpha)$ transforms as a tensor under $SO(n)$,

$$[\alpha_{pq}, f_{rs}(\alpha)] = \delta_{qr} f_{ps}(\alpha) - \delta_{pr} f_{qs}(\alpha) + \delta_{qs} f_{rp}(\alpha) - \delta_{ps} f_{rq}(\alpha). \tag{19}$$

By putting $q = r$ and using $f_{rs}(\alpha) = f_{sr}(\alpha)$, we have

$$\alpha f(\alpha) + \overline{f(\alpha)\alpha} = nf(\alpha) - \mathrm{tr}\,[f(\alpha)],$$

with the help of which the antisymmetry of $g(\alpha)$, defined as in Eq. (17), is readily established. On the other hand, if $f_{rs}(\alpha)$ is replaced in Eq. (19) by $g_{rs}(\alpha)$, where $g_{rs}(\alpha) = -g_{sr}(\alpha)$, we deduce that

$$\alpha g(\alpha) - \overline{g(\alpha)\alpha} = (n-2)g(\alpha),$$

which establishes the symmetry of $h(\alpha)$, defined as in Eq. (18).

Noting that $\alpha^0 = 1$ is symmetric and that α is antisymmetric, we see from these results that any polynomial of degree $2l$ in α, $l = 0, 1, 2, \ldots$, can be expressed as the sum of a symmetric one of degree $2l$ and an antisymmetric one of *lower* degree. Similarly, any polynomial of degree $(2l + 1)$ can be expressed as the sum of an antisymmetric one of that degree and a symmetric one of lower degree.

We then infer from Eqs. (11)–(14) that $F_n(\alpha)$ is symmetric or antisymmetric, according as n is even or odd, i.e.,

$$\overline{F_n(\alpha)} = (-1)^n F_n(\alpha). \tag{20}$$

4. METHOD FOR THE DETERMINATION OF THE MATRIX POLYNOMIAL

For any given value of n, $F_n(\alpha)$ can be calculated by reduction to polynomial form of the appropriate equation (A3) or (A10), as shown in Appendix A for $n = 3, 4$, and 5. However, as n increases, such a calculation quickly becomes very involved. Here we present, in each of the cases n even and n odd, a method of obtaining $F_n(\alpha)$ quite easily for any given n.

(a) When $n = 2m$ is even, $F_n(\boldsymbol{\alpha})$ is completely determined by the conditions

(i) $F_n(\alpha) = G_m(a^2)$, where $G_m(a^2)$ is a polynomial in a^2 of degree m and $a = \alpha - m + 1$,

(ii) $\overline{F_n(\alpha)} = F_n(\alpha)$,

(iii) $\mathrm{tr}\,[F_n(\alpha)] = 0$,

which follow from Eqs. (12), (20), and (11). The proof is as follows.

Consider the sequence of polynomials defined by

$$f_0 = 1, \quad f_1 = \alpha(\alpha - m + 1) + b_0 f_0,$$

$$f_{l+1} = [\alpha(\alpha - 2m + 1) + b_l]f_l + [\tfrac{1}{2}\alpha + c_l\alpha(\alpha - m + 1) + d_l]\,\mathrm{tr}\,[f_l],$$

$$l = 1, 2, \ldots, \tag{21}$$

where b_0, b_l, c_l, and d_l are arbitrary constants. According to the results of the preceding section, each polynomial in the sequence is symmetric.

Moreover, in view of the conditions (ii) and (iii) above, we see that, for some choice of b_0, b_l, c_l, and d_l, $l = 1, 2, \ldots, m-1$,

$$F_n = f_m - \mathrm{tr}\,[f_m]/(2m). \tag{22}$$

Then f_m, like F_n, must be even in a, and we shall use this to determine b_0, b_l, c_l, and d_l uniquely [except for d_{m-1}, which remains arbitrary, but which does not in any way contribute to F_n, in view of Eq. (22)].

We denote by f_l^0 that part of f_l which is a linear combination of positive powers of α, with numerical coefficients, i.e., not involving $\sigma_2, \sigma_3, \ldots$. It is evident that

$$f_l^0 = \alpha(\alpha - m + 1)p_l[\alpha(\alpha - 2m + 1)]$$
$$= a(a + m - 1)p_l[(a + m - 1)(a - m)],$$

where $p_l(x)$ is a polynomial of the $(l-1)$th degree in x. It is also clear that if f_m is even in a, so are f_m^0 and $(f_m - f_m^0)$. But, if f_m^0 is even in a, we have

$$(a + m - 1)p_m[(a + m - 1)(a - m)]$$
$$= (a - m + 1)p_m[(a - m + 1)(a + m)].$$

We set, in succession, $a = m - 1$, $a = m - 2, \ldots$, $a = 1$ in this identity and thus obtain (for $n > 2$)

$$p_m[1(2 - 2m)] = p_m[2(3 - 2m)] = \ldots$$
$$= p_m[(m - 1)(-m)] = 0,$$

and therefore

$$p_m(x) = (x + 2m - 2)(x + 4m - 6) \cdots$$
$$\times [x + m(m - 1)],$$

$$f_m^0 = (a + m - 1)[(a - m + 1)(a + m - 2)]$$
$$\times [(a - m + 2)(a + m - 3)] \cdots [(a - 1)(a)]a,$$
$$= \alpha[(\alpha - 2m + 2)(\alpha - 1)][(\alpha - 2m + 3)(\alpha - 2)]$$
$$\times \cdots [(\alpha - m)(\alpha - m + 1)](\alpha - m + 1).$$

Thus $b_l = l(2m - l + 1)$, $l = 0, 1, \ldots, m - 1$, and in order that f_m should also be an even function of a, we must take $c_l = -1/(2l)$, $l = 1, 2, \ldots, m - 1$ and $d_l = -\frac{1}{2}(m - 1)$, $l = 1, 2, \ldots, m - 2$, while d_{m-1} is left arbitrary.

Thus, when $n = 2m$,

$$F_n(\alpha) = f_m - \mathrm{tr}\,[f_m]/(2m),$$

where

$$f_1 = \alpha(\alpha - m + 1),$$

$$f_{l+1} = (\alpha - l)(\alpha - 2m + l + 1)f_l - (\alpha - m + 1)$$
$$\times (\alpha - l)\,\mathrm{tr}\,[f_l]/(2l). \tag{23}$$

For example, when $n = 2$,

$$F_2(\alpha) = f_1 - \tfrac{1}{2}\,\mathrm{tr}\,[f_1],$$
$$f_1 = \alpha^2;$$

for $n = 4$ [cf. Eq. (A6)],

$$F_4(\alpha) = f_2 - \tfrac{1}{4}\,\mathrm{tr}\,[f_2],$$
$$f_1 = \alpha(\alpha - 1),$$
$$f_2 = (\alpha - 2)(\alpha - 1)^2 - \tfrac{1}{2}(\alpha - 1)^2\sigma_2;$$

for $n = 6$,

$$F_6(\alpha) = f_3 - \tfrac{1}{6}\,\mathrm{tr}\,[f_3],$$
$$f_1 = \alpha(\alpha - 2),$$
$$f_2 = (\alpha - 4)(\alpha - 2)(\alpha - 1)\alpha - \tfrac{1}{2}(\alpha - 2)(\alpha - 1)\sigma_2,$$
$$f_3 = (\alpha - 4)(\alpha - 3)(\alpha - 2)^2(\alpha - 1)\alpha - \tfrac{1}{2}(\alpha - 3)$$
$$\times (\alpha - 2)^2(\alpha - 1)\sigma_2 - \tfrac{1}{8}(\alpha - 2)^2[2\sigma_4 - 14\sigma_3$$
$$+ 16\sigma_2 - (\sigma_2)^2].$$

(b) When $n = 2m + 1$ is odd, $F_n(\alpha)$ is completely determined by the conditions

(i) $F_n(\alpha) = (a - \frac{1}{2})G_m(a^2)$, where $G_m(a^2)$ is a polynomial in a^2 of degree m and $a = \alpha - m + \frac{1}{2}$,

(ii) $\overline{F_n(\alpha)} = -F_n(\alpha)$,

which follow from Eqs. (13) and (20).

In this case we consider the sequence of antisymmetric polynomials generated by

$$g_1 = \alpha,$$
$$g_{l+1} = [\alpha(\alpha - 2m) + b_l]g_l + [c_l\alpha + \tfrac{1}{2}]\,\mathrm{tr}\,[\alpha g_l],$$
$$l = 1, 2, \ldots .$$

For some choice of b_l and c_l, $l = 1, 2, \ldots, m$, we must have $F_n = g_{m+1}$. We determine these constants uniquely by requiring that g_{m+1} satisfy condition (i) above.

Suppose g_l^0 is obtained from g_l by dropping terms involving σ_k. Then

$$g_{l+1}^0 = \alpha p_l[\alpha(\alpha - 2m)],$$
$$= [a + m - \tfrac{1}{2}]p_l[(a - \tfrac{1}{2})^2 - m^2],$$

where $p_l(x)$ is of degree l in x. From condition (i), we see that g_{m+1}^0 must vanish for $a = \frac{1}{2}$, so that $p_m(-m^2) = 0$ for $m > 0$, and we can write

$$p_m(x) = (x + m^2)q_m(x),$$
$$g_{m+1}^0 = (a + m - \tfrac{1}{2})(a - \tfrac{1}{2})^2 q_m[(a - \tfrac{1}{2})^2 - m^2].$$

Again from condition (i), we see that $(a+m-\frac{1}{2})(a-\frac{1}{2})q_m[(a-\frac{1}{2})^2-m^2]$ must be an even polynomial in a, so that

$$(a+m-\tfrac{1}{2})(a-\tfrac{1}{2})q_m[(a-\tfrac{1}{2})^2-m^2]$$
$$=(a-m+\tfrac{1}{2})(a+\tfrac{1}{2})q_m[(a+\tfrac{1}{2})^2-m^2].$$

We set, in succession, $a=m-\frac{1}{2}$, $a=m-\frac{3}{2}$, $\ldots$, $a=\frac{3}{2}$, and obtain, for $m>1$,

$$q_m[1(1-2m)]=q_m[2(2-2m)]=\cdots$$
$$=q_m[(m-1)(-1-m)]=0,$$

whence

$$q_m(x)=[x+1(2m-1)][x+2(2m-2)]\cdots$$
$$\times[x+(m-1)(m+1)],$$

$$g^0_{m+1}=(a-\tfrac{1}{2})(a+m-\tfrac{1}{2})[(a-m+\tfrac{1}{2})(a+m-\tfrac{3}{2})]$$
$$\times[(a-m+\tfrac{3}{2})(a+m-\tfrac{5}{2})]\cdots$$
$$\times[(a-\tfrac{3}{2})(a+\tfrac{1}{2})](a-\tfrac{1}{2}),$$
$$=\alpha[(\alpha-2m)+1(2m-1)][\alpha(\alpha-2m)+2(2m-2)]$$
$$\times\cdots[\alpha(\alpha-2m)+m^2].$$

For $m=1$, $q_m=1$. Thus $b_l=l(2m-l)$, and, to make $g_{m+1}/(a-\frac{1}{2})$ an even polynomial in a, we must also take $c_l=-1/(2l)$.

Thus, when $n=2m+1$,

$$F_n(\alpha)=g_{m+1}(\alpha),$$

where

$$g_1=\alpha,$$
$$g_{l+1}=(\alpha-l)(\alpha-2m+l)g_l-(\alpha-l)\operatorname{tr}[\alpha g_l]/(2l). \tag{24}$$

For example, when $n=3$ [cf. Eq. (A12)],

$$g_2=(\alpha-1)^2\alpha-\tfrac{1}{2}(\alpha-1)\sigma_2;$$

for $n=5$ [cf. Eq. (A13)],

$$g_2=(\alpha-1)(\alpha-3)\alpha-\tfrac{1}{2}(\alpha-1)\sigma_2,$$
$$g_3=(\alpha-2)^2(\alpha-3)(\alpha-1)\alpha-\tfrac{1}{2}(\alpha-1)(\alpha-2)^2\sigma_2$$
$$-\tfrac{1}{4}(\alpha-2)[\sigma_4-4\sigma_3+3\sigma_2-\tfrac{1}{2}(\sigma_2)^2].$$

5. INVARIANTS OF $SO(n)$

Consider the sequence of antisymmetric polynomials $e_l(\alpha)$, $l=1,2,\ldots$, defined by

$$e_1=\alpha,\quad e_{l+1}=\alpha(\alpha-n+1)e_l+\tfrac{1}{2}\operatorname{tr}[\alpha e_l].$$

The identity $\operatorname{tr}[e_l]=0$ then expresses σ_{2l-1} as a function of $\sigma_2,\sigma_3,\ldots,\sigma_{2l-2}$. Thus

$$\sigma_3=\tfrac{1}{2}(n-2)\sigma_2,$$
$$\sigma_5=\tfrac{1}{2}(3n-4)\sigma_4-\tfrac{1}{2}(n-1)(n-2)\sigma_3-\tfrac{1}{2}(\sigma_2)^2,$$

etc.

We see also, from the fact that $\operatorname{tr}[\alpha^k F_n(\alpha)]=0$, $k=1,2,\ldots$, that $\sigma_{n+1},\sigma_{n+2},\ldots$ can be expressed as functions of $\sigma_2,\sigma_3,\ldots,\sigma_n$. It follows that all σ_k can be regarded as functions of σ_{2l}, $l=1,2,\ldots,m$, or, alternatively, and more conveniently from our point of view, as functions of τ_l, where

$$\tau_l=\operatorname{tr}[f_l]/(2l),\qquad n=2m,$$
$$\tau_l=\operatorname{tr}[\alpha g_l]/(2l),\qquad n=2m+1,$$

with f_l, g_l as in Eqs. (23) and (24).

The functional dependence of τ_l on Λ_i, $i,l=1,2,\ldots,m$, and hence the eigenvalues of τ_l, can be determined by comparing the two forms $G_m(a^2)$ given, on the one hand in Eq. (14), on the other via Eqs. (23) or (24).

Thus in the case $n=2m+1$, we deduce from Eq. (24) that

$$G_m(a^2)=\{-\tau_m-[a^2-\tfrac{1}{4}]\tau_{m-1}$$
$$-[a^2-\tfrac{1}{4}][a^2-\tfrac{9}{4}]\tau_{m-2}-\cdots$$
$$-[a^2-\tfrac{1}{4}][a^2-\tfrac{9}{4}]\cdots[a^2-(m-\tfrac{3}{2})^2]\tau_1$$
$$+[a^2-\tfrac{1}{4}][a^2-\tfrac{9}{4}]\cdots[a^2-(m-\tfrac{1}{2})^2]\}, \tag{25}$$

so that

$$\tau_m=-G_m(\tfrac{1}{4}),\qquad \tau_{m-1}=\tfrac{1}{2}[-\tau_m-G_m(\tfrac{9}{4})],$$
$$\tau_{m-2}=[-\tau_m-6\tau_{m-1}-G_m(\tfrac{25}{4})]/24,\qquad \text{etc.}$$

Then from Eq. (14) we have, writing $\chi_i=(\Lambda_i+\frac{1}{2}n-i)^2$,

$$\tau_m=(-1)^{m+1}\prod_{i=1}^m(\chi_i-\tfrac{1}{4}),$$
$$\tau_{m-1}=(-1)^{m+1}\left(\prod_{i=1}^m(\chi_i-\tfrac{9}{4})-\prod_{i=1}^m(\chi_i-\tfrac{1}{4})\right),\qquad \text{etc.}$$

We note also, from a comparison of Eqs. (25) and (A10), that

$$\beta_p^{(m)}\beta_p^{(m)}=-m\tau_m=m(-1)^m\prod_{i=1}^m(\chi_i-\tfrac{1}{4}).$$

In the case $n=2m$, we find from Eq. (23) that

$$G_m(a^2)=\{-\tau_m-a^2\tau_{m-1}-a^2[a^2-1]\tau_{m-2}-\cdots$$
$$-a^2[a^2-1]\cdots[a^2-(m-2)^2]\tau_1$$
$$+a^2[a^2-1]\cdots[a^2-(m-1)^2]\}, \tag{26}$$

$$\tau_m = -G_m(0), \quad \tau_{m-1} = -\tau_m - G_m(1),$$

$$\tau_{m-2} = \tfrac{1}{12}[-\tau_m - 4\tau_{m-1} - G_m(4)], \quad \text{etc.}$$

Again writing $\chi_i = (\Lambda_i + \frac{1}{2}n - i)^2$, we see from Eq. (14) that

$$\tau_m = (-1)^{m+1} \prod_{i=1}^{m} \chi_i,$$

$$\tau_{m-1} = (-1)^{m+1}\left[\prod_{i=1}^{m} (\chi_i - 1) - \prod_{i=1}^{m} \chi_i\right], \quad \text{etc.}$$

The set of operators $\tau_1, \tau_2, \ldots, \tau_{m-1}$ and τ_m (or $\beta_p^{(m)}\beta_p^{(m)}$) is a complete set of invariants for $SO(2m+1)$. However, in the case $n = 2m$, as is well known, one pseudoscalar invariant such as $\beta^{(m)}$ (see Appendix A) or Λ_m is needed. We note from a comparison of Eqs. (26) and (A3) that $[\beta^{(m)}]^2/m^2 = -\tau_m$, so that

$$\beta^{(m)} = m(i)^m \prod_{i=1}^{m} (\Lambda_i + \tfrac{1}{2}n - i),$$

where we have taken the sign of the square root which is consistent with that weight vector interpretation of $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ adopted in Appendix B. The set of operators τ_l, $l = 1, 2, \ldots, m-1$, and $\beta^{(m)}$ is a complete set of invariants for $SO(2m)$.

APPENDIX A:

Here we establish the existence of an nth degree polynomial identity for α. Following Bakri,[15] we define the completely antisymmetric tensors $\beta^{(k)}_{pq\cdots t}$, $k = 0, 1, \ldots, m$, of rank $n - 2k$, with

$$\beta^{(0)}_{pq\ldots v} = \epsilon_{pq\ldots v},$$

$$\beta^{(k)}_{pq\ldots t} = \tfrac{1}{2}\alpha_{uv}\beta^{(k-1)}_{pq\ldots tuv}. \tag{A1}$$

Then one finds that, for $k = 0, 1, \ldots, m-1$,[15]

$$\beta^{(k)}_{pq\ldots tu}\alpha_{uv} - k\beta^{(k)}_{pq\ldots tv} = (-1)^{n+1}(\delta_{pv}\beta^{(k+1)}_{qr\ldots t} - \delta_{qv}\beta^{(k+1)}_{pr\ldots t} + \cdots + (-1)^n\delta_{tv}\beta^{(k+1)}_{pq\ldots s})/(k+1). \tag{A2}$$

In the case $k = 0$, this identity is proved by inspection, and the proof for general values of k is obtained by induction.

(a) When $n = 2m$, we take $k = m-1$ in Eq. (A2) to obtain

$$\beta^{(m-1)}_{pq}\alpha_{qr} - (m-1)\beta^{(m-1)}_{pr} = -\delta_{pr}\beta^{(m)}/m,$$

i.e., $\beta^{(m-1)}(\alpha - m + 1) = -\beta^{(m)}/m$. Moreover, $\alpha\beta^{(m-1)} = \beta^{(m-1)}\alpha$, so that

$$(\alpha - m + 1)^2[\beta^{(m-1)}]^2 = [\beta^{(m)}]^2/m^2. \tag{A3}$$

The quantity $[\beta^{(m-1)}]^2$ can be reduced to a polynomial of degree $(2m-2)$ in α, so that Eq. (A3) is in fact the required identity. It is not a simple matter to complete this reduction for a general value of m. However, for small values of m, the desired result can be obtained from the identity

$$\epsilon_{pij\ldots klr}\epsilon_{qst\ldots uvr} = \sum(-1)^{S(P)}\delta_{pq}\delta_{is}\delta_{jt}\cdots\delta_{ku}\delta_{lv}, \tag{A4}$$

where $\Sigma(-1)^{S(P)}$ means a sum over all permutations of $p, i, \ldots, k$, and l, with appropriate signatures, by multiplying with

$$(\alpha_{ij}\cdots\alpha_{kl})(\alpha_{st}\cdots\alpha_{uv}) \tag{A5}$$

and shuffling factors till the required order is reached. For example, when $n = 4$,

$$[\beta^{(1)}]^2 = -\overline{\alpha^2} + \tfrac{1}{2}\sigma_2 = -\alpha^2 + 2\alpha + \tfrac{1}{2}\sigma_2$$

so that

$$(\alpha - 2)(\alpha - 1)^2\alpha - \tfrac{1}{2}(\alpha - 1)^2\sigma_2 + \tfrac{1}{4}[\beta^{(2)}]^2 = 0. \tag{A6}$$

{In the important related case of $SO(3,1)$, with generators $J_{\lambda\mu}$, $\lambda, \mu = 0, 1, 2, 3$, satisfying

$$[J_{\lambda\mu}, J_{\rho\sigma}] = -i(g_{\lambda\rho}J_{\mu\sigma} + g_{\mu\sigma}J_{\lambda\rho} - g_{\mu\rho}J_{\lambda\sigma} - g_{\lambda\sigma}J_{\mu\rho})$$

where $g_{\lambda\mu}$ is the pseudo-Euclidean metric tensor, the corresponding result is

$$J_\mu{}^\alpha J_\alpha{}^\rho J_\rho{}^\sigma J_\sigma{}^\nu - 4i\, J_\mu{}^\alpha J_\alpha{}^\rho J_\rho{}^\nu + (J_1 - 5)\, J_\mu{}^\alpha J_\alpha{}^\nu - 2i(J_1 - 1)J_\mu{}^\nu = (J_1 + [J_2]^2)\delta_\mu{}^\nu,$$

where $J_1 = \frac{1}{2}J_{\lambda\mu}J^{\lambda\mu}$, and $J_2 = \frac{1}{8}\epsilon_{\lambda\mu\nu\rho}J^{\lambda\mu}J^{\nu\rho}$.}

(b) When $n = 2m+1$, we take $k = m-1$ in Eq. (A2) to obtain

$$\beta^{(m-1)}_{pqr}\alpha_{rs} - (m-1)\beta^{(m-1)}_{pqs} = (1/m)(\delta_{sp}\beta^{(m)}_q - \delta_{sq}\beta^{(m)}_p). \tag{A7}$$

Premultiplying this equation, on the one hand by α_{pq}, on the other by $\beta^{(m)}_p$, we obtain

$$\alpha_{pq}\beta^{(m)}_q = \beta^{(m)}_q\alpha_{qp} = m\beta^{(m)}_p, \tag{A8}$$

and

$$\gamma_{qr}\alpha_{rs} + (m-1)\gamma_{sq} = (1/m)(\beta^{(m)}_s\beta^{(m)}_q - \beta^{(m)}_p\beta^{(m)}_p\delta_{qs}), \tag{A9}$$

where $\gamma_{pq} = -\gamma_{qp} = \beta^{(m)}_r\beta^{(m-1)}_{rpq}$. Next, premultiplying Eq. (A9) by $(\alpha_{us} - m\delta_{us})$ and using Eq. (A8), we obtain

$$(\alpha - m)[\overline{\gamma\alpha} + (m-1)\gamma + (1/m)\beta_p^{(m)}\beta_p^{(m)}] = 0$$

or, since γ is antisymmetric,

$$(\alpha - m)[(\alpha - m)\gamma + (1/m)\beta_p^{(m)}\beta_p^{(m)}] = 0. \quad \text{(A10)}$$

This is the required identity, as γ can be reduced to a polynomial of degree $(n-2)$ in α.

In this case, using Eq. (A4), we find

$$\gamma_{pq} = 2^{(2-n)}(\alpha_{ij} \cdots \alpha_{kl})(\alpha_{st} \cdots \alpha_{uv}) \times \sum (-1)^{S(P)} \delta_{pq}\delta_{is}\delta_{jt} \cdots \delta_{kn}\delta_{lv}. \quad \text{(A11)}$$

For example, when $n = 3$, we find $\gamma = \alpha$, so that

$$(\alpha - 1)[(\alpha - 1)\alpha + \beta_p^{(1)}\beta_p^{(1)}] = 0; \quad \text{(A12)}$$

for $n = 5$, after a lengthy calculation, we find

$$\gamma = 2\alpha^3 - 8\alpha^2 + (6 - \sigma_2)\alpha + \sigma_2,$$

so that

$$(\alpha - 2)\{(\alpha - 2)[2\alpha^3 - 8\alpha^2 + (6-\sigma_2)\alpha + \sigma_2] + \tfrac{1}{2}\beta_p^{(2)}\beta_p^{(2)}\} = 0. \quad \text{(A13)}$$

APPENDIX B:

Here we justify the assertion that an arbitrary n-vector operator θ can be resolved into components satisfying Eqs. (5).

An irreducible representation of $SO(n)$ is characterized by a set of integers or half-odd integers $(\lambda_1, \lambda_2 \; \ldots \;, \lambda_m)$, as in Eq. (4). This set may be interpreted as the weight vector of highest weight for the corresponding representation,[16] when each weight vector is defined as an ordered set of eigenvalues of, in particular, $-i\alpha_{12}, -i\alpha_{34}, \ldots, -i\alpha_{2m-1,2m}$. Accordingly, for such a representation, one can find an element ψ of the representation space, such that

$$-i\alpha_{2i-1,2i}\psi = \lambda_i\psi, \quad i = 1, 2, \ldots, m. \quad \text{(B1)}$$

Moreover, since ψ corresponds to the highest weight and since, for $q > 2i > 2j$,

$$(-i\alpha_{2i-1,2i})(\alpha_{2i-1,q} + i\alpha_{2i,q}) = (\alpha_{2i-1,q} + i\alpha_{2i,q})(-i\alpha_{2i-1,2i} + 1)$$

and

$$[(-i\alpha_{2j-1,2j}), (\alpha_{2i-1,q} + i\alpha_{2i,q})] = 0,$$

it follows that

$$(\alpha_{2i-1,q} + i\alpha_{2i,q})\psi = 0, \quad q > 2i. \quad \text{(B2)}$$

One sees conversely that any ψ satisfying Eqs. (B1) and (B2) belongs to the representation labeled $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. For example, Eq. (B2) implies that (no summation over repeated subscripts here except where indicated)

$$(\alpha_{2i-1,q} - i\alpha_{2i,q})(\alpha_{2i-1,q} + i\alpha_{2i,q})\psi = 0, \quad q > 2i,$$

i.e.,

$$(\alpha_{2i-1,q}\alpha_{q,2i-1} + \alpha_{2i,q}\alpha_{q,2i})\psi = -i\alpha_{2i-1,2i}\psi = \lambda_i\psi.$$

Thus

$$\sum_{q=2i+1}^{n} (\alpha_{2i-1,q}\alpha_{q,2i-1} + \alpha_{2i,q}\alpha_{q,2i}) = (n-2i)\lambda_i\psi,$$

$$\left[\sum_{q=2i}^{n} (\alpha_{2i-1,q}\alpha_{q,2i-1}) + \sum_{q=2i+1}^{n} (\alpha_{2i,q}\alpha_{q,2i})\right]\psi = \lambda_i(\lambda_i + n - 2i)\psi,$$

and

$$\sigma_2\psi = \sum_{i=1}^{m} 2\lambda_i(\lambda_i + n - 2i)\psi.$$

Now suppose there exists an n-vector operator θ acting on ψ. In the case $n = 2m+1$, define $\psi_i^{\pm} = (\theta_{2i-1} \pm i\theta_{2i})\psi$, $i = 1, 2, \ldots, m$, and $\psi^0 = \theta_n\psi$. It follows from Eqs. (2), (B1), and (B2) that $\psi_i^{\pm}$ and ψ^0 correspond to weight vectors $(\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i \pm 1, \lambda_{i+1}, \ldots, \lambda_m)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and that the set $\psi^0, \psi_i^{\pm}$, $i = 1, 2, \ldots, m$ is invariant under the action of the operators $(\alpha_{2j-1,q} + i\alpha_{2j,q})$, $j = 1, 2, \ldots, m$, $q > 2j$.

We deduce therefore that $\theta\psi$ has components only in respresentations labeled $(\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i \pm 1, \lambda_{i+1}, \ldots, \lambda_m)$ $i = 1, 2, \ldots, m$, or $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. It is easy to see then that the same must be true if ψ is replaced by any element of the representation space labeled $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, so that the required result follows for $n = 2m+1$. The argument for the case $n = 2m$ is similar except that ψ^0 does not occur, and $\theta\psi$ can have no component in the representation labeled $(\lambda_1, \lambda_2, \ldots, \lambda_m)$.

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1 P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon, Oxford, 1930).

2 P. Güttinger and W. Pauli, Z. Phys. **67**, 743 (1931).

3 E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Braunschweig, 1931; English Ed., Academic, New York, 1959).

4 G. Racah, Phys. Rev. **62**, 438 (1942).

5 E. V. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge U.P., Cambridge, 1935).

[6] J. C. Slater, *The Quantum Theory of Atomic Structure* (Mc-Graw-Hill, New York, 1960), Vol. II.
[7] G. Racah, in *Ergebnisse der exacten Naturwissenschaften* (Springer, Berlin, 1965), Vol. 37.
[8] L. C. Biedenharn, in *Lectures on Theoretical Physics, Boulder 1962* (Interscience, New York, 1963), Vol. 5. See also L. C. Biedenharn and J. D. Louck, Commun. Math. Phys. **8**, 89 (1968).
[9] J. D. Louck, Los Alamos Scientific Laboratory Report LA2451 (1960).
[10] H. J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945); **21**, 451 (1949).
[11] W. Killing, Math. Ann. **31**, 252 (1888); **33**, 1 (1889); **34**, 57 (1889); **36**, 161 (1890).
[12] Y. Lehrer-Ilamed, Bull. Res. Council Israel **5A**, 197 (1956).
[13] Y. Lehrer-Ilamed, in *Spectroscopic and Group Theoretical Methods in Physics*, edited by F. Bloch *et al.* (North-Holland, Amsterdam, 1968). See also H. Goldberg and Y. Lehrer-Ilamed, J. Math. Phys. **4**, 501 (1963); B. Gruber and L. O' Raifeartaigh, *ibid.* **5**, 1796 (1964).
[14] G. Racah, in *Group Theoretical Concepts and Methods in Elementary Particle Physics* (Lectures of the 1962 Istanbul Summer School), edited by F. Gürsey, (Gordon and Breach, New York, 1964).
[15] M. M. Bakri, Commun. Math. Phys. **9**, 146 (1968).
[16] F. D. Murnaghan, *The Theory of Group Representations* (Dover, New York, 1963).

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Characteristic Identities for Generators of $GL(n)$, $O(n)$ and $Sp(n)$

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A hierarchy of tensor identities, satisfied by the generators of the general linear group $GL(n)$, is obtained in terms of two different sets of invariants. An application to the identification of irreducible representations and the decomposition of reducible representations is described. Similar results are obtained for the generators of orthogonal, pseudo-orthogonal, and symplectic groups.

1. INTRODUCTION

The generators of the group $GL(n)$ satisfy the commutation relations[1]

$$[a^i_{\ j}, a^k_{\ l}] = \delta^k_{\ j} a^i_{\ l} - \delta^i_{\ l} a^k_{\ j}, \quad i,\ldots,l = 1,\ldots,n, \tag{1}$$

and their matrix representations are of some interest, as they also furnish representations for other Lie algebras. Indeed, the commutation relations

$$[z_i, z_j] = C^k_{\ ij} z_k,$$

in which the structure constants necessarily satisfy

$$C^k_{\ ij} + C^k_{\ ji} = 0, C^l_{\ ij}C^m_{\ kl} + C^l_{\ jk}C^m_{\ il} + C^l_{\ ki}C^m_{\ jl} = 0,$$

can be satisfied identically by writing

$$z_i = C^k_{\ ji} a^j_{\ k},$$

provided that $a^i_{\ j}$ satisfy (1). Different irreducible representations of the $a^i_{\ j}$ are, moreover, associated with different sets of eigenvalues of the invariants σ_r defined by (repeated affixes $i, j, k, l, \ldots$ are understood to be summed over values from 1 to n; however, subscripts $r, s, \ldots$ are exempted from this summation convention)

$$\sigma_1 = a^i_{\ i}, \quad \sigma_2 = a^i_{\ j} a^j_{\ i}, \quad \sigma_3 = a^i_{\ j} a^j_{\ k} a^k_{\ i}, \tag{2}$$

etc., which are Casimir operators, i.e., commute with all elements of the algebra. Thus, an irreducible representation of $GL(n)$ can, in principle, be identified by determining the eigenvalues of $\sigma_1, \sigma_2, \ldots, \sigma_n$. Of course, such a representation is not necessarily irreducible for the z_i, and the σ_r, $r \leqslant n$, are not necessarily independent. The problem of determining a complete set of independent invariants for a semisimple Lie algebra has been considered by Biedenharn[2] and by Gruber and O'Raiffeartaigh.[3]

Another, less explicit but sometimes more convenient way of defining a set of invariants for $GL(n)$ is in terms of the highest weights of the finite-dimensional irreducible representations. Let λ_1 be an operator whose eigenvalue, in a particular representation R of this kind, is the same as the maximum eigenvalue ℓ_1 of $a^1_{\ 1}$ in this representation. Further, let $\lambda_r, r = 2, \ldots, n)$ be an operator whose eigenvalue ℓ_r in R is the same as the maximum eigenvalues of $a^r_{\ r}$, when $a^1_{\ 1}, \ldots, a^{r-1}_{\ \ r-1}$ have the eigenvalues $\ell_1, \ldots, \ell_{r-1}$, respectively. Then, if ψ is a vector such that $a^r_{\ r}\psi = \ell_r\psi$, it must satisfy $a^i_{\ j}\psi = 0, j > i$, and (as one can verify by computing $\sigma_1\psi$ and $\sigma_2\psi$)

$$\sigma_1 = \sum_{r=1}^{n} \lambda_r, \quad \sigma_2 = \sum_{r=1}^{n} \lambda_r(\lambda_r + n + 1 - 2r). \tag{3}$$

The representation R is labeled by $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$, where $\ell_r - \ell_s$ is integral and nonnegative when $r < s$, and can be identified in this way if the dependence of the first n of the σ_r on the λ_s is known. Unfortunately, the complexity of the expressions for the σ_r in terms of the λ_s increases rapidly with r.

One use of the invariants is in the decomposition of a reducible representation into distinct irreducible components, which can be solved by determining the projections on to different eigenvectors of the σ_r (or λ_s). There are, of course, other ways of dealing with this problem, notably the method of character analysis, which has been applied to $U(n)$ and $SU(n)$ by Blaha.[4] Our present interest in the problem arises from its connection with a