Bound states, resonances, and symmetries of a neutral Dirac particle with anomalous magnetic moment, coupled to a fixed monopole

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For the system consisting of a neutral Dirac particle with anomalous magnetic moment, interacting with a fixed magnetic monopole, zero-energy bound states are constructed for each possible value of the total angular momentum. Results of Kazama and Yang for the charge–monopole system are used to deduce the existence of other bound states for this system, when the mass of the bound particle is nonzero. In the zero-mass case, there are no other bound states, but there are resonant states, and these are determined exactly. A noncompact, so(3,2) symmetry algebra of the zero-energy bound states is given for the finite-mass case and for the zero-mass case. In each case the infinite number of such states is associated with an irreducible Majorana representation of the algebra.

I. INTRODUCTION

This work continues the study of bound states and resonances of relativistic two-body problems involving magnetic interactions. Earlier related works have discussed a relativistic charged particle in a monopole field,[1] a Dirac particle with anomalous magnetic moment in a Coulomb field,[2] a neutrino with anomalous moment in a Coulomb field,[3] a Dirac particle in the field of a magnetic dipole,[4] and various other charge–dipole models.[5] The general forms of relativistic potentials describing the interactions between charges, magnetic monopoles, and magnetic dipoles (i.e., anomalous magnetic moments) have been derived from field theory.[6] Some of the techniques necessary to handle the singular potentials which can arise in such problems have been developed.[7]

The problem considered here is that of a neutral spin-$\frac{1}{2}$ particle having mass m and anomalous magnetic moment a, interacting with a fixed monopole having magnetic charge g. It is assumed that a relativistic description of the particle in an external electromagnetic field $F^{\mu\nu}$ is provided by Dirac's equation with a Pauli coupling term $ia\gamma_\mu\gamma_\nu F^{\mu\nu}$, in the usual notation, so that the Hamiltonian for the system under discussion is

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta(m + \theta\boldsymbol{\sigma}\cdot\mathbf{r}/r^3), \tag{1.1}$$

where $\theta = 2ag$ and the remaining symbols have their usual meanings. [We use $\boldsymbol{\sigma}$ to denote the Pauli matrices and also the 4×4 matrices

$$\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix},$$

relying on context to fix the meaning of a particular usuage.]

Concerning physical applications, it is questionable to what extent the study of this H gives realistic information about the bound states and resonances of a *neutron* interacting with a fixed monopole. In particular, at the short distances which characterize the strongly bound (zero-energy) states of H discussed below (if m and a are given the values appropriate to the neutron, and $|eg| = \frac{1}{2}$, where e is the electronic charge), the strong field of the monopole would certainly detect the extended structure of the neutron, so that H might define a poor approximation to the true dynamics. However, the problem may be of interest at the substructure level of particles. Furthermore, the problem is of considerable indirect interest because H is sufficiently simple to permit an exact determination of some (though not all) bound states when $m > 0$; and of bound states and resonances in the limiting case $m = 0$. (If it should be found that one or more of the neutrinos has a magnetic moment, this latter limit could become physically relevant.) The Hamiltonian H therefore defines a simple relativistic model, in which some of the characteristic features of magnetic interactions[5] can be determined exactly. The mathematical interest of the system is increased by the remarkable appearance of a noncompact Lie algebra so(3,2) as an invariance algebra of the infinite set of bound states associated with the eigenvalue $E = 0$ of H. This occurs whether or not $m = 0$.

The system with Hamiltonian H may be regarded as the special case $Z = 0$ of the charge–monopole system considered by Kazama, Yang, and Goldhaber,[8–11] who took

$$H' = \boldsymbol{\alpha}\cdot(\mathbf{p} - Ze\mathbf{A}) + \beta(m + \theta\boldsymbol{\sigma}\cdot\mathbf{r}/r^3), \tag{1.2}$$

where $\mathbf{A}$ is the monopole potential. However, it must be noted that the structure of the conserved angular momentum vector, and indeed the possible values of the total angular momentum quantum number j, are quite different in cases with $eg \neq 0$ ($j = |eg| - \frac{1}{2}, |eg| + \frac{1}{2},\ldots$) and $eg = 0$ ($j = \frac{1}{2}, \frac{3}{2},\ldots$). In the latter case, which is the one of interest here, the singular monopole potential does not appear in H and the angular momentum therefore has the familiar form

$$\mathbf{J} = \mathbf{r}\wedge\mathbf{p} + \tfrac{1}{2}\boldsymbol{\sigma} = \mathbf{L} + \tfrac{1}{2}\boldsymbol{\sigma}. \tag{1.3}$$

Kazama and Yang[9,10] found bound states of two types for H'. Type A occur for $j \geqslant |eg| + \frac{1}{2}$, and type B for $j = |eg| - \frac{1}{2}$. They found eigenfunctions of H' corresponding to eigenvalue $E = 0$, and hence to binding energies equal to m, for every j (types A and B). They also showed that there are infinitely many other bound states of type B for $|\theta|$ sufficiently large, and found numerical estimates of some of the corresponding eigenvalues. Later Olaussen *et al.*[12] found approximate analytic expressions for these type B eigenfunctions and eigenvalues. In the meantime, Yang[11] showed that there exists an

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infinite number of bound states of type A for each j value, such that

$$|eg| + \tfrac{1}{2} \leqslant j < (|eg|^2 + 2m|\theta| + \tfrac{1}{4})^{1/2} - \tfrac{1}{2}. \quad (1.4)$$

The type B bound states do not occur for the Hamiltonian H. However, the "radial part" of the eigenvalue problem for H has the same structure as that for the type A bound states of H', so we are able to adapt relevant results of Kazama and Yang to the case we consider.

The problem of determining resonant states of H in the case $m > 0$ remains intractable, but in the limit $m = 0$ we find that the radial problem reduces to one of the cases described by Barut *et al.*[7] for which exact solutions can be given, corresponding to bound states and resonances.

The coupling of a spin-$\frac{1}{2}$ particle (with magnetic moment) to a fixed monopole has been considered in various nonrelativistic approximations.[13] The variety of results obtained for binding energies reflects the fact that the nonrelativistic Hamiltonian with attractive $\boldsymbol{\sigma}\cdot\mathbf{r}/r^3$ potential is not essentially self-adjoint, and an *ad hoc* repulsive core or cutoff has to be introduced to regularize the eigenvalue problem. In contrast, the relativistic problem, whether for a charged or uncharged particle, has an effective potential which already has a repulsive $1/r^4$ core, and needs no regularization. The nonrelativistic approaches typically miss the strongly bound, zero-energy states found by Kazama *et al.* and ourselves, which are characterized by distances at which the nature of the core is critical.

II. BOUND STATES WHEN $m > 0$

Introducing the Hermitian matrix γ_5 ($= i\alpha_1\alpha_2\alpha_3$), we note that $i\beta\gamma_5$ anticommutes with H. It is therefore convenient to adopt a representation of the Dirac matrices in which this matrix is diagonal. We take

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & -i\boldsymbol{\sigma} \\ i\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad (2.1)$$

where I is the 2×2 unit matrix, so that

$$i\beta\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (2.2)$$

The bispinor Ψ is now written as

$$\psi = \begin{pmatrix} f \\ g \end{pmatrix}, \quad (2.3)$$

where f and g have two components each and the eigenvalue equation $H\psi = E\psi$ becomes

$$\begin{aligned} &(-i\boldsymbol{\sigma}\cdot\mathbf{p} + \theta\sigma_r/r^2 + m)g = Ef, \\ &(i\boldsymbol{\sigma}\cdot\mathbf{p} + \theta(\sigma_r/r^2) + m)f = Eg, \end{aligned} \quad (2.4)$$

where $\sigma_r = \boldsymbol{\sigma}\cdot\mathbf{r}/r$.

Introducing

$$R = \mathbf{L}\cdot\boldsymbol{\sigma} + 1, \quad (2.5)$$

so that, on f or g,

$$R^2 = \mathbf{J}^2 + \tfrac{1}{4}, \quad \{R, \sigma_r\} = 0, \quad (2.6)$$

we suppose now that ψ is also an eigenvector of $\mathbf{J}^2$ with eigenvalue $j(j+1)$, where $2j$ is a positive integer. Then we can write, in the coordinate representation,

$$\begin{aligned} rf &= f_k(r)\chi_k + f_{-k}(r)\chi_{-k}, \\ rg &= g_k(r)\chi_k + g_{-k}(r)\chi_{-k}, \end{aligned} \quad (2.7)$$

where $f_{\pm k}, g_{\pm k}$ are one-component functions, which vanish at $r = 0$ and are square integrable on $[0, \infty)$; and $\chi_{\pm k}$ are two-component vectors which do not depend on the radial variable r, and which satisfy

$$R\chi_{\pm k} = k\chi_{\pm k}, \quad k = j + \tfrac{1}{2}. \quad (2.8)$$

(Each $\chi_{\pm k}$ can also be labeled by an eigenvalue of J_3; we suppress these labels.) With a suitable choice of phases, we also have

$$\sigma_r\chi_{\pm k} = \pm i\chi_{\mp k}. \quad (2.9)$$

Now

$$\boldsymbol{\sigma}\cdot\mathbf{p} = \sigma_r(p_r + (i/r)R), \quad (2.10)$$

where

$$p_r = \tfrac{1}{2}(r^{-1}\mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}r^{-1}). \quad (2.11)$$

On $f_{\pm k}$ and $g_{\pm k}$, p_r equals $-i(d/dr)$, so that Eqs. (2.4) give

$$\begin{aligned} &(d_r + k/r - \theta/r^2)g_{-k} = img_k - iEf_k, \\ &(d_r - k/r - \theta/r^2)g_k = -img_{-k} + iEf_{-k}, \\ &(d_r + k/r + \theta/r^2)f_{-k} = -imf_k + iEg_k, \\ &(d_r - k/r + \theta/r^2)f_k = imf_{-k} - iEg_{-k}, \end{aligned} \quad (2.12)$$

where $d_r = d/dr$.

Suppose for definiteness that $\theta > 0$; the treatment when $\theta < 0$ is quite similar and the corresponding results are obtained by interchanging the roles of f_k and g_k, and of f_{-k} and $-g_{-k}$. Consider first the case $E = 0$. Then Eqs. (2.12) fall into two uncoupled pairs.

The first pair is

$$\begin{aligned} &(d_r + k/r - \theta/r^2)g_{-k} = img_k, \\ &(d_r + k/r - \theta/r^2)g_k = -img_{-k}, \end{aligned} \quad (2.13)$$

and implies

$$\left(d_r^2 - \frac{2\theta}{r^2} - \frac{k(k-1)}{r^2} + \frac{2\theta}{r^3} + \frac{\theta^2}{r^4}\right)g_k = m^2 g_k. \quad (2.14)$$

This has the acceptable solution (square integrable, and vanishing at $r = 0$)

$$\begin{aligned} g_k &= A_k \sum_{l=0}^{k-1} \frac{(k+l-1)!}{l!(k-l-1)!}(2mr)^{-l}e^{-\theta/r - mr} \\ &= A_k(2mr/\pi)^{1/2}K_{k-1/2}(mr)e^{-\theta/r}, \end{aligned} \quad (2.15)$$

where A_k is a constant to be determined by normalization, and K_j is a modified Bessel function.[14] The second of Eqs. (2.13) then gives[14]

$$g_{-k} = -iA_k(2mr/\pi)^{1/2}K_{k+1/2}(mr)e^{-\theta/r}. \quad (2.16)$$

The second pair of Eqs. (2.12) implies

$$\left(d_r^2 + \frac{2\theta}{r^2}d_r - \frac{k(k-1)}{r^2} - \frac{2\theta}{r^3} + \frac{\theta^2}{r^4}\right)f_k = m^2 f_k, \quad (2.17)$$

which is also solvable in terms of Bessel functions, but has no acceptable nontrivial solutions. Thus f_k and from the last of Eqs. (2.12), f_{-k}, must vanish.

The (unnormalized) eigenfunctions of H and $\mathbf{J}^2$, with eigenvalues 0 and $j(j+1) = k^2 - \frac{1}{4}$, now follow from Eqs. (2.3) and (2.7). Because $f_{\pm k} = 0$, it follows from Eqs. (2.2) and (2.3) that these eigenfunctions are also eigenvectors of $i\beta\gamma_5$, with eigenvalue -1. (When $\theta < 0$, the corresponding eigenfunctions have $i\beta\gamma_5 = +1$.) They are of essentially the zame form as the zero-energy eigenfunctions found by Kazama and Yang,[9,10]

The determination of the nonzero eigenvalues of H is much more difficult, and exact solutions have not been found. We have to deal with all four coupled equations (2.12). However, if we put (for the case $\theta > 0$)

$$r = \theta\rho, \quad A_0 = \theta m, \quad B_0 = \theta E,$$
$$h_1 = \tfrac{1}{2}(f_k + g_k), \quad h_2 = -\tfrac{1}{2}i(f_{-k} + g_{-k}), \tag{2.18}$$
$$h_3 = \tfrac{1}{2}(f_k - g_k), \quad h_4 = -\tfrac{1}{2}i(f_{-k} - g_{-k}),$$

we obtain precisely the equations considered by Kazama and Yang (with their μ replaced by our k). We may then adapt the qualitative results obtained by Yang.[11]

Thus, for any value of j such that $j(j+1) < 2m\theta$, there is an infinite sequence of eigenvalues of H, which is bounded below, and bounded above by m; and there is an image set which is bounded above, and bounded below by $-m$. No estimates of the eigenvalues are available, and the forms of the corresponding eigenfunctions are unknown. For any value of j such that $j(j+1) > 2m\theta$, there are no nonzero eigenvalues of H. If it should happen that $2m\theta$ equals $j(j+1)$ for some j, then $+m$ and $-m$ are the only nonzero eigenvalues of H for that value of j.

It may be expected that, as in the $m = 0$ case (see Section III), the Hamiltonian H with $m > 0$ also exhibits resonances in Gamow's sense.[15] These would correspond to complex values of E for which Eqs. (2.12) admit a solution in which $f_{\pm k}$ $g_{\pm k}$ vanish at $r = 0$ and behave like $\exp(i\lambda r)$ as $r \to \infty$, where $\lambda^2 = E^2 - m^2$. The problem of finding these resonance values E, and the corresponding functions, has not been solved. In fact it is difficult to see how the analysis sketched by Yang, for (real) nonzero eigenvalues, could be extended to give even qualitative results about the existence of resonances.

III. BOUND STATES AND RESONANCES WHEN $m = 0$

In this case Eqs. (2.12) reduce to two uncoupled pairs of coupled equations. Taking $\theta > 0$, and setting $r = \theta\rho$, we have

$$\begin{aligned}
(d_\rho + k/\rho - 1/\rho^2)g_{-k} &= -i\lambda f_k,\\
(d_\rho - k/\rho + 1/\rho^2)f_k &= -i\lambda g_{-k},\\
(d_\rho - k/\rho - 1/\rho^2)g_k &= i\lambda f_{-k},\\
(d_\rho + k/\rho + 1/\rho^2)f_{-k} &= i\lambda g_k,
\end{aligned} \tag{3.1}$$

where $d_\rho = d/d\rho$ and $\lambda = \theta E$.

Consider first the zero-energy bound states ($\lambda = 0$), which are seen to be associated now with uncoupled first-order equations. These integrate to give

$$\begin{aligned}
g_{-k} &= A\rho^{-k}e^{-1/\rho}, \quad f_k = B\rho^k e^{1/\rho},\\
g_k &= C\rho^k e^{-1/\rho}, \quad f_{-k} = D\rho^{-k}e^{-1/\rho},
\end{aligned} \tag{3.2}$$

where A, B, C, and D are constants. Solutions which behave acceptably at $\rho = 0$ and $\rho = \infty$ are obtained only if $B = C = D = 0$. It follows from Eqs. (2.3) and (2.7) that the resulting eigenfunctions of H and J^2 are also eigenfunctions of R and $i\beta\gamma_5$, with eigenvalues $-k$ and -1, respectively. (When $\theta < 0$, these eigenvalues become $-k$ and $+1$.) It may be noted that with increasing angular momentum (increasing k), these eigenfunctions become more and more concentrated near $r = 0$.

Are there any nonzero eigenvalues of H? If λ is real and nonzero, then it can be seen from Eqs. (3.1) that for large ρ, each of $g_{\pm k}, f_{\pm k}$ behaves like

$$ce^{i\lambda\rho} + de^{-i\lambda\rho}, \tag{3.3}$$

with c and d constant, and is therefore not normalizable. The answer is therefore no.

We now seek solutions corresponding to resonant states, by allowing λ complex in Eqs. (3.1) and requiring that $g_{\pm k}, f_{\pm k}$ vanish at $\rho = 0$ and behave like $\exp(i\lambda\rho)$ as $\rho \to \infty$. We can suppose $\lambda \neq 0$, since that case has been discussed. The first pair of Eqs. (3.1) gives

$$\begin{aligned}
\left(d_\rho^2 - \frac{k(k+1)}{\rho^2} + \frac{2(k+1)}{\rho^3} - \frac{1}{\rho^4}\right)g_{-k} &= -\lambda g_{-k},\\
\left(d_\rho^2 - \frac{k(k-1)}{\rho^2} + \frac{2(k-1)}{\rho^3} - \frac{1}{\rho^4}\right)f_k &= -\lambda^2 f_k,
\end{aligned} \tag{3.4}$$

while the second pair gives

$$\begin{aligned}
\left(d_\rho^2 - \frac{k(k-1)}{\rho^2} - \frac{2(k-1)}{\rho^3} - \frac{1}{\rho^4}\right)g_k &= -\lambda^2 g_k,\\
\left(d_\rho^2 - \frac{k(k-1)}{\rho^2} - \frac{2(k+1)}{\rho^3} - \frac{1}{\rho^4}\right)f_k &= -\lambda^2 f_{-k}.
\end{aligned} \tag{3.5}$$

Consider the last equation. It is the same as the radial equation we would obtain for a nonrelativistic spinless particle with total angular momentum k, mass $\frac{1}{2}$ and energy λ^2, moving in the purely repulsive central potential

$$V(\rho) = 2(k+1)/\rho^3 + 1/\rho^4, \tag{3.6}$$

which decreases monotonically for $\rho > 0$. It is evident that there are no resonances for such a potential, and so f_{-k} must vanish. It then follows from the last of Eqs. (3.1) that $g_k = 0$.

Now consider the first of Eqs. (3.4), which is of the form considered by Barut *et al.*[7] Following their analysis, we seek a solution which behaves like $\exp(-1/\rho)$ as $\rho \to 0$, and like $\exp(i\lambda\rho)$ as $\rho \to \infty$. Setting

$$g_{-k} = G(\rho)\rho^{-k}\exp(-1/\rho + i\lambda\rho), \tag{3.7}$$

we find

$$\left[d_\rho^2 + \left(\frac{2}{\rho^2} - \frac{2k}{\rho} + 2i\lambda\right)d_\rho + \left(\frac{2i\lambda}{\rho^2} - \frac{2ik\lambda}{\rho}\right)\right]G = 0. \tag{3.8}$$

We now seek a solution of this equation which is a polynomial in ρ of finite degree. [This is a sufficient, though possibly not necessary, condition that g_{-k} as in Eq. (3.7) will have the right behavior as $\rho \to \infty$.] Then we find that this degree must equal k, that is,

$$G(\rho) = \sum_{n=0}^{k} A_n\rho^n. \tag{3.9}$$

Substitution of this expression in Eq. (3.8) leads to the condition

$$
W\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{k-1} \\ A_k \end{pmatrix}
\equiv \begin{pmatrix}
i\lambda & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-i\lambda k & i\lambda - k & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & -i\lambda(k-1) & i\lambda + 1 - 2k & 3 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdots & -2i\lambda & i\lambda - \frac{1}{2}(k-1)(k+2) & k \\
0 & 0 & 0 & 0 & \cdots & 0 & -i\lambda & i\lambda - \frac{1}{2}k(k+1)
\end{pmatrix}
\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_{k-1} \\ A_k \end{pmatrix} = 0, \tag{3.10}
$$

for the coefficients A_n, and hence to the condition

$$\det W = 0, \tag{3.11}$$

for λ. Equation (3.11) is a polynomial equation of the $(k+1)$th degree in λ. The polynomial always has λ^2 as a factor, corresponding to the zero-energy case already discussed, where g_{-k} takes the form in Eqs. (3.2) and only A_0 in Eq. (3.9) is nonzero. The complex roots always appear in pairs λ, $-\lambda^*$, each such pair corresponding to a single resonance at the energy value $E \sim \text{Re}\,\lambda$, with width $\sim \text{Im}\,\lambda$.

For example, when $k = 1$,

$$\det W = -\lambda^2, \tag{3.12}$$

so there are no resonances in this case—only the zero-energy bound state.

When $k = 2$,

$$\det W = -\lambda^2(i\lambda - 1), \tag{3.13}$$

and there is a resonance (antibound state) with $\lambda = -i$.

When $k = 3$,

$$\det W = -\lambda^2(-\lambda^2 - 4i\lambda + 6), \tag{3.14}$$

and there is a resonance with $\lambda = \pm\sqrt{2} - 2i$.

For $k = 4$ and 5, calculation of exact values for the roots of Eq. (3.11) is possible but increasingly complicated. For $k \geqslant 6$, we are faced with a polynomial equation of degree greater than or equal to five, and must resort to numerical methods to find the roots to any desired accuracy.

Once a nonzero root λ has been determined, the corresponding coefficents A_n can be found from Eq. (3.10), and hence g_{-k} determined from Eqs. (3.7) and (3.9). Then f_k is given by the first of Eqs. (3.1); it has the form

$$f_k = F(p)\rho^{-(k-1)} \exp(-1/\rho + i\lambda\rho), \tag{3.15}$$

where F is a polynomial of degree $(k-1)$. We could alternatively find f_k by applying the analysis of Barut *et al.*[7] directly to the second of Eqs. (3.4).

Because f_{-k} and g_k vanish for all these resonances, the latter correspond, according to Eqs. (2.3) and (2.7), to eigenfunctions of $i\beta\gamma_5 R$ with eigenvalue $+1$. (For $\theta < 0$, the corresponding eigenvalue is -1.) It may be noted that $i\beta\gamma_5 R$ is a constant of the motion when $m = 0$ but not when $m > 0$.

IV. ZERO-ENERGY BOUND STATES AND so(3,2) SYMMETRY

It is convenient to introduce the Hermitian operators

$$\mathbf{n} = \mathbf{r}/r, \quad \mathbf{e} = \tfrac{1}{2}(\mathbf{n} \wedge \mathbf{L} - \mathbf{L} \wedge \mathbf{n}), \tag{4.1}$$

which satisfy

$$
\begin{aligned}
&[n_i, n_j] = 0, \quad [e_i, e_j] = -i\epsilon_{ijk} L_k, \\
&[e_i, n_j] = i(\delta_{ij} - n_i n_j), \\
&e_i n_j - e_j n_i = \epsilon_{ijk} L_k, \\
&\mathbf{e}\cdot\mathbf{n} = i = -\mathbf{n}\cdot\mathbf{e},
\end{aligned} \tag{4.2}
$$

and then to define

$$
\begin{aligned}
\mathbf{u} &= R\mathbf{n} + i\mathbf{e} = \mathbf{n}R + i\mathbf{e} - i\boldsymbol{\sigma} \wedge \mathbf{n}, \\
\mathbf{d} &= \mathbf{u}^\dagger = \mathbf{n}R - i\mathbf{e} = R\mathbf{n} - i\mathbf{e} + i\boldsymbol{\sigma} \wedge \mathbf{n}.
\end{aligned} \tag{4.3}
$$

It is then straightforward to check that

$$
\begin{aligned}
&R\mathbf{u} = \mathbf{u}(R+1), \quad R\,\mathbf{d} = \mathbf{d}(R-1), \\
&[u_i, u_j] = 0 = [d_i, d_j], \quad [u_i, d_j] = -2R\,\delta_{ij} - 2i\,\epsilon_{ijk} J_k.
\end{aligned} \tag{4.4}
$$

In addition, we have, of course,

$$[u_i, J_j] = i\,\epsilon_{ijk} u_k, \quad [d_i, J_j] = i\,\epsilon_{ijk} d_k, \quad [R, J_i] = 0. \tag{4.5}$$

If we set, for $i, j = 1, 2, 3$,

$$
\begin{aligned}
&l_{ij} = \epsilon_{ijk} J_k, \quad l_{45} = R, \\
&l_{i4} = -l_{4i} = \tfrac{1}{2}(u_i + d_i), \\
&l_{i5} = -l_{5i} = \tfrac{1}{2}i(u_i - d_i),
\end{aligned} \tag{4.6}
$$

then relations (4.4) and (4.5) can be written in a standard form for so(3,2),

$$[l_{AB},l_{CD}] = i(g_{AC}l_{BD} + g_{BD}l_{AC} - g_{BC}l_{AD} - g_{AD}l_{BC}), \tag{4.7}$$

where A, B, C, D run over 1 to 5, and the metric tensor $g_{AB} = \text{diag}(1,1,1,-1,-1)$. Note that each l_{AB} is Hermitian.

These operators all commute with the radial variables r and ρ_r, and can be thought of as acting in the vector space spanned by the χ_k and χ_{-k}, $k = 1, 2,\ldots$. In fact, they leave invariant separately the subspaces S_+ and S_- spanned by the χ_k and χ_{-k}, respectively, since none of the l_{AB} change the sign of R. (Because R does not have zero as an eigenvalue, it follows that the lowering operator **d** annihilates χ_1, and the raising operator **u** annihilates χ_{-1}.) On each of S_+ and S_- the l_{AB} span an irreducible Majorana representation of so(3,2) which remains irreducible when restricted to so(3,1). To see this, take as so(3,1) basis operators the l_{ij} and l_{i4}, and note that the two so(3,1) invariants have the form

$$\begin{aligned} C_1 &\equiv \tfrac{1}{2}l_{ij}l_{ij} - l_{i4}l_{i4} \\ &= \mathbf{J}^2 - \tfrac{1}{4}(\mathbf{u}^2 + \mathbf{d}^2 + \mathbf{u}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{u}), \qquad (4.8) \\ C_2 &\equiv \epsilon_{ijk}l_{ij}l_{k4} = \mathbf{J}\cdot\mathbf{u} + \mathbf{J}\cdot\mathbf{d}. \end{aligned}$$

It follows from the definitions (4.3) that

$$\begin{aligned} &\mathbf{u}^2 = 0 = \mathbf{d}^2, \quad \mathbf{u}\cdot\mathbf{d} = (2R-1)(R-1), \\ &\mathbf{d}\cdot\mathbf{u} = (2R+1)(R+1), \quad \mathbf{j}\cdot\mathbf{u} = 0 = \mathbf{j}\cdot\mathbf{d}. \end{aligned} \tag{4.9}$$

Therefore, noting the first of Eqs. (2.6), we have

$$C_1 = -\tfrac{3}{4}, \quad C_2 = 0. \tag{4.10}$$

In the infinite-dimensional irreducible representation $[k_0,c]$ of so(3,1), where j takes values k_0, $k_0+1,\ldots$, these two invariants equal[16] $(k_0^2 + c^2 - 1)$ and $2i\,k_0c$, respectively, and it follows that we are dealing with the representation $[\frac{1}{2},0]$. It is well known[17] that this extends in two different ways to irreducible Majorana representations of so(3,2). In one of these, say R_+, l_{45} takes positive eigenvalues; in the other, say R_-, it takes negative eigenvalues. We see that we have the representation $R\pm$ on $S\pm$. Note that the operator σ_r intertwines these two representations, according to the second of Eqs. (2.6), and that

$$\mathbf{u}\,\sigma_r = -\sigma_r\mathbf{d}, \quad \mathbf{d}\,\sigma_r = -\sigma_r\mathbf{u}. \tag{4.11}$$

The Casimir operator of so(3,2) has the same value on $R_\pm$. In fact, it is given in each case by

$$\begin{aligned} &\tfrac{1}{2}l_{ij}l_{ij} - l_{i4}l_{i4} - l_{i5}l_{i5} + (l_{45})^2 \\ &\quad = -\tfrac{3}{4} + \tfrac{1}{4}(\mathbf{u}^2 + \mathbf{d}^2 - \mathbf{u}\cdot\mathbf{d} - \mathbf{d}\cdot\mathbf{u}) + R^2 = -\tfrac{5}{4}, \end{aligned} \tag{4.12}$$

using Eqs. (4.9).

Because this so(3,2) algebra is independent of the radial variables, it underlies all problems involving Dirac's equation (or even the Schrödinger–Pauli equation for a spin-$\frac{1}{2}$ particle) with a centrally symmetric potential or external field. However, it is not an invariance algebra of any Hamiltonian of such an equation, in general, and indeed it is not the invariance algebra of the zero-energy eigenspace of the Hamiltonian H of Eq. (1.1), whether or not $m = 0$.

To define the latter invariance algebra, consider first the case $m = 0$. Introduce (with $\theta > 0$ and $r = \theta\rho$ as before)

$$\mathbf{U} = \rho\mathbf{u}, \quad \mathbf{D} = \rho^{-1}\mathbf{d}, \tag{4.13}$$

which can be seen to satisfy the same relations among themselves, and with R and $\mathbf{J}$, as the **u** and **d**. [See Eqs. (4.4), (4.5), and (4.9).] Then the generators L_{AB}, defined in terms of **U**, **D**, R, and **J** just as the l_{AB} were defined in terms of **u**, **d**, R, and **J**, also span a representation of so(3,2). The L_{AB} can be seen to leave invariant the vector space P_0 spanned by vectors of the form

$$\psi_k = \begin{pmatrix} 0 \\ \rho^{-(k+1)}e^{-1/\rho}\chi_{-k} \end{pmatrix}, \tag{4.14}$$

which is just the zero-energy eigenspace of H in the case $m = 0$, $\theta > 0$ [cf. Eqs. (2.3) and (2.2)]. The invariance of this subspace can also be seen from the relations

$$\begin{aligned} &\{H,R\} = 0, \quad [H,\mathbf{J}] = 0, \\ &H\mathbf{U} = -\rho^2\mathbf{D}H, \quad H\mathbf{D} = -\rho^{-2}\mathbf{U}H, \end{aligned} \tag{4.15}$$

which hold when $m = 0$, and which are established with the help of Eqs. (4.4) and (4.11).

Note that **U** is not a Hermitian conjugate to **D** as defined, so the L_{AB} are not Hermitian. Nevertheless, they satisfy on P_0 exactly the same relations as do the l_{AB} on S_-, spanning an irreducible Majorana representation R_- there. They are related by a complicated similarity transformation to Hermitian operators on P_0.

In the case $m > 0$, the zero-energy eigenfunctions of H are not eigenfunctions of R, but rather of

$$|R| = \epsilon R, \tag{4.16}$$

where ϵ is the operator with eigenvalue ± 1 on each $\chi_{\pm k}$, and hence on $S_\pm$. With $\theta > 0$, we define

$$\begin{aligned} \mathbf{U}' &= \frac{1}{m}\mathbf{u}\left(ip_r - \frac{R}{r} - \frac{\theta}{r^2}\right)\frac{1}{2}(1+\epsilon) + \frac{1}{m}\mathbf{d}\left(ip_r + \frac{R-1}{r} - \frac{\theta}{r^2}\right)\frac{1}{2}(1-\phi), \\ \mathbf{D}' &= \frac{1}{m}\mathbf{d}\left(ip_r + \frac{R-1}{r} - \frac{\theta}{r^2}\right)\frac{1}{2}(1+\epsilon) + \frac{1}{m}\mathbf{u}\left(ip_r - \frac{R}{r} - \frac{\theta}{r^2}\right)\frac{1}{2}(1-\epsilon). \end{aligned} \tag{4.17}$$

Then

$$\begin{aligned} [U'_i,U'_j] = {} & \frac{1}{m^2}[u_i,u_j]\frac{1}{2}(1+\epsilon)\left(ip_r - \frac{R+1}{r} - \frac{\theta}{r^2}\right)\left(ip_r - \frac{R}{r} - \frac{\theta}{r^2}\right) + \frac{1}{m^2}[d_i,d_j] \\ & \times\frac{1}{2}(1-\epsilon)\left(ip_r + \frac{R-2}{r} - \frac{\theta}{r^2}\right)\left(ip_r + \frac{R-1}{r} - \frac{\theta}{r^2}\right) = 0, \end{aligned} \tag{4.18}$$

and similarly

$$[D'_i,D'_j] = 0. \tag{4.19}$$

Also,

$$m^2 U_i' D_j' = u_i d_j \frac{1}{2}(1+\epsilon)\Big(ip_r - \frac{R-1}{r} - \frac{\theta}{r^2}\Big)\Big(ip_r + \frac{R-1}{r} - \frac{\theta}{r^2}\Big) + d_i u_j \frac{1}{2}(1-\epsilon)$$
$$\times\Big(ip_r + \frac{R}{r} - \frac{\theta}{r^2}\Big)\Big(ip_r - \frac{R}{r} - \frac{\theta}{r^2}\Big). \tag{4.20}$$

With $g_k(r)$ as in Eq. (2.15), we have on $g_k\chi_k$,

$$\Big(ip_r - \frac{R-1}{r} - \frac{\theta}{r^2}\Big)\Big(ip_r - \frac{R-1}{r} - \frac{\theta}{r^2}\Big) = \Big(d_r^2 - \frac{2\theta}{r^2} d_r - \frac{k(k-1)}{r^2} + \frac{2\theta}{r^3} + \frac{\theta^2}{r^4}\Big) = m^2, \tag{4.21}$$

so that the first term on the right-hand side of Eq. (4.20) reduces to

$$m^2 d_i u_j \tfrac{1}{2}(1+\epsilon). \tag{4.22}$$

In a similar way, the second term reduces on $g_{-k}\chi_{-k}$ [with g_{-k} as in Eq. (2.16)], to

$$m^2 d_i u_j \tfrac{1}{2}(1-\epsilon). \tag{4.23}$$

Then, on

$$\psi = \begin{pmatrix} 0 \\ r^{-1}(g_k\chi_k + g_{-k}\chi_{-k}) \end{pmatrix}, \tag{4.24}$$

we have

$$U_i' D_j' = u_i d_j \tfrac{1}{2}(1+\epsilon) + d_i u_j \tfrac{1}{2}(1-\epsilon), \tag{4.25}$$

and similarly

$$D_j' U_i' = d_j u_i \tfrac{1}{2}(1+\epsilon) + u_j d_i \tfrac{1}{2}(1-\epsilon), \tag{4.26}$$

so that

$$[U_i', D_j'] = [u_i, d_j]\tfrac{1}{2}(1+\epsilon) - [u_j, d_i]\tfrac{1}{2}(1-\epsilon)$$
$$= -2|R|\delta_{ij} - 2i\epsilon_{ijk} J_k. \tag{4.27}$$

Using the same kind of manipulations we check that on such functions ψ, **U**, **D**, R, and **J** also satisfy the remaining relations (4.4), (4.5), and (4.9) satisfied by **u**, **d**, R, and **J**.

Defining L'_{AB} by analogy with L_{AB}, we have, on the space P'_0 of vectors spanned by ψ of the form (4.24), an irreducible R_+ Majorana representation of so(3,2).

Like L_{AB}, the operators L'_{AB} are not Hermitian, but in a similar way they are related by a similarity transformation to Hermitian operators on the zero-energy eigenspace P'_0.

In closing this section, we remark that the irreducible Majorana representations of SO(3,2) are known to be integrable to unitary representations of the group SO(3,2) (or more accurately, of its double-covering group.)

V. CONCLUDING REMARKS

The appearance of zero-energy bound states for this system, as for the charge–monopole system considered by Kazama *et al.*, is remarkable. Such zero-energy modes are also found for Dirac particles interacting with non-Abelian, su(2) monopoles,[18] suggesting that their occurrence may have a topological interpretation. The operator $i\beta\gamma_5$ is diagonal on all the zero-energy states found by Kazama *et al.* and ourselves. The corresponding solutions $\psi(\mathbf{x})$ of Dirac's equation are necessarily static, and it can be seen that they are therefore eigenstates of CT, where C and T are the usual charge conjugation and time-reversal operators, acting on a general spinor [for our representation (2.1) of Dirac matrices] as

$$C\psi(\mathbf{x},t) = \alpha_2\psi^*(\mathbf{x},t),$$
$$T\psi(\mathbf{x},t) = i\alpha_2\beta\gamma_5\psi^*(\mathbf{x},-t),$$

so that

$$CT\,\psi(\mathbf{x}) = i\beta\gamma_5\psi(\mathbf{x}).$$

The other bound states ($m>0$) and resonant states ($m=0$) of H are not eigenstates of CT. Moreover, neither these states nor the zero-energy states are eigenstates of the parity operator

$$P\psi(\mathbf{x},t) = \beta\psi(-\mathbf{x},t),$$

because of the presence of the pseudoscalar σ_r in H.

It is also remarkable that the zero-energy states are associated with a Majorana representation of so(3,2), whether or not $m=0$. We are familiar with accidental symmetries in bound-state problems, associated with finite degeneracies and compact invariance algebras, but the appearance in such problems of infinite degeneracies associated with noncompact algebras is quite unusual. It seems likely that noncompact invariance algebras can also be found for the zero-energy bound states of the charge–monopole system, but we have not pursued that here.

For the Hamiltonian H with $m=0$, we have been able to determine the form of resonant states, occurring at all possible values of the total angular momentum except $j=\frac{1}{2}(k=1)$. Although this system may turn out to be unphysical, it is nevertheless satisfying to have found a relativistic model in which such states can be determined exactly.

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