2006 QAMT Problem-Solving Competition - Year 8 Paper

All questions have equal value.

Question 1
The game of Scrabble gives values to letters: A = 1, B = 3, C = 3, D = 2, E = 1, F = 4, G = 2, H = 4, I = 1, J = 8, K = 5, L = 1, M = 3, N = 1, O = 1, P = 3, Q = 10, R = 1, S = 1, T = 1, U = 1, V = 4, W = 4, X = 8, Y = 4, Z = 10. The value of a word is the sum of the values of the letters. For example, ONE = 1 + 1 + 1 = 3. The name of which positive integer has Scrabble score equal to itself?

Solution
Twelve 12 = 1 + 4 + 1 + 1 + 4 + 1.

Question 2
In triangle ABC, angle ABC is 120°. Point D is chosen in the triangle so that line DA bisects angle BAC and line DC bisects angle BCA. What is the angle ADC?

Solution
From triangle ABC 2x + 2y + 120 = 180, while from triangle ADC x + y + y = 180. Multiplying by 2 gives 2x + 2y + 2α = 360. Combining these gives 2α + 120 = 180, so α = 150.

Question 3
Which fraction x with \( \frac{1}{10} < x < \frac{1}{9} \) has smallest (positive) denominator?

Solution
2/19 Suppose \( \frac{1}{10} < \frac{a}{b} < \frac{1}{9} \) with a, b positive integers. Then 9a < b < 10a. a = 1 is impossible, and a = 2 gives \( \frac{a}{b} = \frac{2}{19} \). If a ≥ 3 then b > 27.

Question 4
What are the last two decimal digits of 61^{2006}?

Solution
61^{5} ends with last two digits 01, so 61^{2006} = 61 \cdot (61^{5})^{401} ends with 61 \cdot (01)^{401} = 61.

Question 5
In front of you is a pile of coins, containing between 1500 and 3000 coins. If you divide it into groups of 7, 11 or 13 each time you have 4 coins left over. How many coins are in the pile?

Solution
2006 The obvious solution is 4 coin, but this impossible since we have at least 100 coins. If we add any multiple of 7 \cdot 11 \cdot 13 = 1001, this does not change the remainder upon dividing by 7, 11 or 13, so 2002 + 4 is a solution. Alternative solution: let n be the number of coins. Then n = 7x + 4 = 11y + 4 for some integers x and y so 7x = 11y so x must be divisible by 11. Thus n = 77z + 4. But also n = 13w + 4 for some w, so now z is divisible by 13, and we get x = 1001t + 4 for some t. To finish with between 1500 and 3000 coins the only possible t we can take is 2, leading to 2006 coins. It is easy to check that this does indeed give a solution.
Question 6
Suppose $x^3 - x - 1 = 0$. Show that $x - 1 = \frac{1}{x^4}$.

Solution First note that $x \neq 0$. Now $x^3 = x + 1$, so $x^4 = x^2 + x$ so $x^5 = x^3 + x^2 = x^2 + x + 1$ so $x^5 - x^4 = 1$. Now divide by $x^4$.

A real number $x > 1$ satisfying $x + 1 = x^m$ and $x - 1 = x^{-n}$ for some $m$ and $n$ is called morphic. Draw a mark on the real line at positions $\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots$. This gives a sort of geometric ruler. The morphic condition is that the sum and difference of consecutive intervals coincide with other intervals. By studying polynomials $x^r - x^s - 1$ it can be shown that the only morphic numbers are the $x \simeq 1.3247$ above and the golden ratio $(\sqrt{5} + 1)/4$. 