2001 UQ/QAMT Competition Solutions, Year 8

1. We can't have $A = 0$ or $E = 0$ since these digits are first digits. In the middle column we have

$$A + R + (carry) \text{ ends in } R$$

So we could have $A = 0$ and no carry or $A = 9$ and carry of 1. Since $A \neq 0$ we must have $A = 9$ and a carry of 1. In the hundreds column, we have

$$E + E + (carry) = A = 9$$

Hence $E = 4$ and there is a carry of 1. Then in the units column $R + 9 = 14$ and hence $R = 5$. Thus we have $E = 4, R = 5$ and $A = 9$.

2. (a) To have teh sum of three digits equal to 24, the smallest possible digit is 6, (since $6 + 9 + 9 = 24$). The first digit of the nubmer could be 6, 7, 8 or 9. A list of all possible numbers is

$$699$$
$$798, 789$$
$$897, 888, 879$$
$$996, 987, 978, 969$$

Thus there are 10 suitable numbers

(b) For the five digit number to be divisible by 4, its last two digits must be divisible by 4. So its last two digits could be 12, 32, 52 or 24. In each case, tehre are three digits left; the first of these could be any one of the three, the second could be either of the two remaining and then there is no choice for the third. So there are $3 \times 2 = 6$ ways of arranging the first three digits, for each of the four possible pairs of last digits. So we ahve $6 \times 4 = 24$ suitable five digit numbers.

3. Let the digits of the credit card be $a, b, c, d, \ldots$ as shown:

$$ab7cde \ldots 8 \ldots$$

Then $a + b + 7 = 18$, $b + 7 + c = 18$, $7 + c + d = 18$, $c + d + e = 18$, and so on. From $a + b + 7 = 18$ and $b + 7 + c = 18$ we get $a = c$. From $b + 7 + c = 18$ and $7 + c + d = 18$ we get $b = d$. Then from $c + d + e = 18$ we see $a + b + e = 18$. But $a + b + 7 = 18$ and so $e = 7$. Thus we have

$$ab7ab7 \ldots 8 \ldots$$

and the pattern repeats in a cyle of three:

$$ab7ab7ab7ab7ab7a$$

Since 8 is the digit third from teh right end, $b = 8$. Then from $a+b+7 = 18$, $a+8+7 = 19$ so $a = 3$. Hence the number is

$$3873873873873873$$

(b) 5 seconds per hour is $5 \times 24/60 = 2$ minutes per day so the silver watch loses 4 minutes per day and the gold watch gains 2 minutes per day. So the gold watch gets 6 minutes ahead of the silver watch each day. Later that week, the gold watch was 33 minutes ahead

of the silver one, so the watches had been going for 5570 days since set correct, that is, for $5\frac{1}{2}$ days.

In $5\frac{1}{2}$ days, the silver watch would have lost $5\frac{1}{2} \times 4 = 22$ minutes. Hence th silver watch was set to the correct time $5\frac{1}{2}$ days less 22 minutes before it showed 7:00. Thus it was set to the correct time at 7:22 on Sunday morning.

4. Trial and error. Start with a table:

Alice's no.	Bill's no.	Carol's no.	Twice Carol's no
100	1008	1100	2200
200	2008	1200	2400
300	3008	1300	2600
400	4008	1400	2800
500	5008	1500	3000
600	6008	1600	3200
700	7008	1700	3400
800	8008	1800	3600
900	9009	1900	3800

This shows that a winning number must be in the 200s. Indeed since Carol's number starts with a 1, twice Carol's number starts with a 2 or a 3. So Bill's number and hence Alice's number must start with a 2 or a 3. But if Alice's number starts with a 3, Carol's number is 13XX, so twice Carol's number starts with a 2 and so can't equal Bill's number.

Alice's no.	Bill's no.	Carol's no.	Twice Carol's no
210	2108	1210	2420
220	2208	1220	2440
230	2308	1230	2460
240	2408	1240	2480
250	2508	1250	2500
260	2608	1260	2520

This shows that a winning number must start with 24 or 25. Indeed since Carol's number starts with 12, twice for Carol's number starts with 24 or 25. If Alice's number if 25X then twie carol's number is $2\times125X = 250X$ or 251X. So only 250 or 251 are possible for a winning number. The table above shows 250 doesn't win and neither does 251 since $2518 \neq 2502 = 2 \times 1251$.

If Alice's number is 24X, then Carol's number is 124X so twice Carol's number is 248X or 249X, so only 248 or 249 are possible winning numbers. $2488 \neq 2496 = 2\times1248$, but $2498 = 2 \times 1249$, so 249, 2498 and 1249 are winning numbers.

5. From the angles in the triangle CDG, angle DHC + 45 + 90 = 180 so angle DHC = 45. From the angles at H, angle EHA + 90 +45 = 180, so angle EHA = 45. Similarly angle AEH = angle FEB = angle BFE = angle GFC = angle FCH = 45. Thus AEH, BFE, GFC and DHC are all isoscelese right angle triangles. So HA = AE = 3 cm and EH = $1.414 \times$ AE = $1.414 \times 3 = 4.242$ cm. Since EFGH is a rectange, FG=EH = 4.242. Then from triange GFC, FC = $1.414 \times$ FG = $1.414 \times 4.242 = 5.998 \approx 6$ cm. Since FC is three times the length of BF, BF = 2cm. Then EB = BF = 2cm, so AB = AE+EB = 3+2 = 5cm, and BC = BF+FC = 2+6 = 8cm. Thus, area ABCD = AB $\times$ BC = $5 \times 8 = 40$ cm^2.