2001 UQ/QAMT Competition Solutions, Year 12

1. Let the three-digit number be abc, so we need $abc = a! + b! + c!$ or

$$100a + 10b + c = a! + b! + c! \qquad (1)$$

We have $0! = 1, 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040, 8! = 40320, 9! = 362880$. Clearly $a, b, c \leq 6$ since otherewise $a!+b!+c! > 5040$. In fact, $a, b, c \leq 5$ since if any of a, b, c were 6, then $a! + b! + c! > 6! = 720$ whereas $100a + 10b + c \leq 666$. Rewrite 1 as

$$100a - a! = b! + c! - (10b + c) \qquad (2)$$

Consider the values of $100a - a!$:

a	$100a - a!$
5	380
4	376
3	294
2	198
1	99

Since $b, c \leq 5$, $b! + c! - (10b + c) < 5! + 5! = 240$. So there is no solution for 2 with $a = 5, 4, 3$. When $a = 2$, 2 becomes $b! + c! - (10b + c) = 198$. Clearly the only hope is with $b = 5$ or $c = 5$. But $b = 5$ gives $c! - c = 198 - 120 + 50 = 128$ with no solution for c and $c = 5$ gives $b! - 10b = 198 - 120 + 5 = 83$, with no solution for b. Thus $a \neq 2$. When $a = 1$, 2 becomes $b! + c! - (10b + c) = 99$, so

$$c! - c = 99 - b! + 10b$$

b	$99 - b! + 10b$	c
0	98	None
1	108	None
2	117	None
3	123	None
4	115	5
5	29	None

Hence the only solution is $a = 1$, $b = 4$, $c = 5$: the only suitable three digit number is 145.

2.

3. Note that $2/(x-1) < 0$ is $x < 1$, $2/(x-1) > 0$ if $x > 1$ an $1/(x - 2 < 0$ if $x < 2$, $1/(x-2) > 0$ if $x > 2$. Thus if $1 < x < 2$ then $[2/(x-1)] < 0$, yet $[2/(x-1)] > 0$ so there is no solution to the equation for $1 < x < 2$.

Consider x with $0 \leq x < 1$. Then $-1 \leq x - 1 < 0$, so $2/(x-1) \leq -2$ and hence $[1/(x-1)] \leq -2$. And $-2 \leq x - 2 < -1$, so $-1 < 1/(x-2) \leq -1/2$, and hence $[1/(x-1) = -1$. Thus $[2/(x-1)] \neq [1/(x-2)]$ for $1 \leq x < 1$. No we can assume $x > 2$, so $2/(x-1) > 0$ and $1/(x-2) > 0$. Take integer n with $n > 0$ and we find when $[2/(x-1)] = n$ and $[1/(x-2)] = n$. To have $[2/(x-1)] = n$ means $n \leq 2/(x-1) \leq n+1$

and so $1/n \geq (x-1)/2 > 1/(n+1)$ and thus $1 + 2/(n+1) < x \leq 1 + 2/n$. Similarly, to have $[1/(x-2] = n$ means $n \leq 1/(x-2) < n+1$ and so $1/n \geq x - 2 > 1/(n+1)$ and thus $2 + 1/(n+1) < x \leq 2 + 1/n$. If $n \geq 2$, $1 + 2/n \leq 2 < 2 + 1/(n+1)$ and these two intervals for x do not overlap. So there are no solutions to the equation for these values of n. If $n = 1$, $[2/(x-1] = 1$ for $2 < x \leq 3$ and $[1/(x-2)] = 1$ for $2\frac{1}{2} < x \leq 3$.

Thus $[2/(x-1)] = [1/(x-2)] = 1$ for $2\frac{1}{2} < x \leq 3$. Thus $[2/(x-1)] = [1/(x-2)] = 1$ for $2\frac{1}{2} < x \leq 3$.

For $n = 0$, we have $[2/(x-1)] = 0$ when $0 \leq 2/(x-1) < 1$ and so $(x-1)/2 > 1$, ie when $x > 3$. And $[1/(x-2)] = 0$ when $0 \leq 1/(x-2) < 1$ and so $x - 2 > 1$, ie when $x > 3$. Then $[2/(x-1)] = [1/(x-2)] = 0$ for $x > 3$.

Thus we find that for $x \geq 0$, $[2/(x-1)] = [1/(x-2)]$ just when $x \geq 2\frac{1}{2}$.

4.

5. (a) Start by drawing up a table

y	$y/2$	$1/(1+y)$
1/2	1/4	1/(1+1/2) = 2/3
1/4	1/8	1/(1+1/4) = 4/5
2/3	1/3	1/(1+2/3) = 3/5
1/8	1/16	1/(1+1/8) = 8/9
4/5	2/5	1/(1+4/5) = 5/9

All the numbers in the table are in C, thus 5/9 is in C.

(b) To show that 15/37 is in C, try to work backwards: 15/37 could have come from $2 \times 15/37 = 30/37$ or x where $1/(1+x) = 15/37$, ie $1+x = 37/15$, so $x = 37/15 - 1 = 22/15$. But 22/15 is not between 0 and 1, so 22/15 is not in C. So if 15/37 is in C, it must have come from 30/37. Can we show 30/37 is in C? We continue like this. It is easiest to make a table. For each number y in the first columns, y could have come from $2y$ or from x where $1/(1+x) = y$, ie $x = 1/y - 1$.

y	$2y$	$1/y - 1$	$0 < 2y < 1$?	$0 < 1/y - 1 < 1$?
14/15	28/15	14/14 - 1 = 1/14	N	Y
1/14	1/7	14/1 - 1 = 13	Y	N
1/7				
2/7				
4/7				
3/4	...	...	...	...
1/3				
2/3				

Since we are told that 1/2 is in C, tracing back through the table shows that 15/37 is also in C.

(c) As in (b), a number y in C could come from $2y$ or from x where $1/(1_x) = y$, ie $x = 1/y - 1$. Now (i) If $0 < y < 1/2$ then $1/y > 2$, so $1/y - 1 > 1$ and hence $1/y - 1$ cannot be in C. So if $0 < y < 1/2$ and y is in C then y must have come from $2y$ (and note that $0 < 2y < 1$). (ii) If $1/2 < y < 1$ then $2y > 1$ so $2y$ cannot be in C. So if $1/2 < y < 1$

then y must have come from $1/y - 1$ (and note when $1/2 < y < 1$ then $2 > 1/y > 1$ so $0 < 1/y - 1 < 1$).

Given any number y with $0 < y < 1$, form a sequence $y_0, y_1, y_2, \ldots$ (as in the table for 15/37), where

$$\begin{aligned} y_0 &= y \\ y_{n+1} &= 2y_n \text{ if } 0 < y_n < \frac{1}{2} \\ y_{n+1} &= \frac{1}{y_n} - 1 \text{ if } \frac{1}{2} < y_n < 1 \end{aligned}$$

We want to show that at some stage m, we have $y_m = 1/2$ (for then tracing back from y_m to y_0 shows that y_0 is in C). Suppose we have z with $1/2 < z < 1$ where $z = p/q$ and suppose p and q have no common factor. Then $1/z - 1 = q/p - 1 = (q-p)/p$, and $q - p$ and p will have no common factor. Note that the new denominator p is smaller than the old demonimator q. So in the sequence $y_0, y_1, \ldots$ defined above whenever the rule $y_{n+1} = 1/y_n - 1$ is used, the denominator of y_{n+1} is smaller than the demonimator of y_n.

An whenever the rule $y_{n+1} = 2y_n$ is used, teh denominator of y_{n+1} is the same as the denominator of y_n (if y_n has odd denominator) or decreases (if y_n has even denominator). With denominators decreasing, teh rule $y_{n+1} = 1/y_n - 1$ can be used only finitely many times (if $y = y_0 = p/q$ this rule can be used at most q times). Suppose the last time it is used is to get y_k. Then $y_{k+1} = 2y_k$, $y_{k+2} = 2y_{k+1} = 2^2 y_k, \ldots$. Since always $y_{n+1} < 1$, this process must stop. (If $1/2^l < y_k$, this doubling rule can be applied at most l times.) It stops only when we reach m with $y_m = \frac{1}{2}$.