2001 UQ/QAMT Competition Solutions, Year 11

1. Let the three-digit number be abc, so we need $abc = a! + b! + c!$ or

$$100a + 10b + c = a! + b! + c! \tag{1}$$

We have $0! = 1, 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040, 8! = 40320, 9! = 362880$. Clearly $a, b, c \leq 6$ since otherewise $a!+b!+c! > 5040$. In fact, $a, b, c \leq 5$ since if any of a, b, c were 6, then $a! + b! + c! > 6! = 720$ whereas $100a + 10b + c \leq 666$. Rewrite 1 as

$$100a - a! = b! + c! - (10b + c) \tag{2}$$

Consider the values of $100a - a!$:

a	$100a - a!$
5	380
4	376
3	294
2	198
1	99

Since $b, c \leq 5$, $b! + c! - (10b + c) < 5! + 5! = 240$. So there is no solution for 2 with $a = 5, 4, 3$. When $a = 2$, 2 becomes $b! + c! - (10b + c) = 198$. Clearly the only hope is with $b = 5$ or $c = 5$. But $b = 5$ gives $c! - c = 198 - 120 + 50 = 128$ with no solution for c and $c = 5$ gives $b! - 10b = 198 - 120 + 5 = 83$, with no solution for b. Thus $a \neq 2$. When $a = 1$, 2 becomes $b! + c! - (10b + c) = 99$, so

$$c! - c = 99 - b! + 10b$$

b	$99 - b! + 10b$	c
0	98	None
1	108	None
2	117	None
3	123	None
4	115	5
5	29	None

Hence the only solution is $a = 1$, $b = 4$, $c = 5$: the only suitable three digit number is 145.

2. (a) Start by drawing up a table

y	$y/2$	$1/(1+y)$
1/2	1/4	1/(1+1/2) = 2/3
1/4	1/8	1/(1+1/4) = 4/5
2/3	1/3	1/(1+2/3) = 3/5
1/8	1/16	1/(1+1/8) = 8/9
4/5	2/5	1/(1+4/5) = 5/9

All the numbers in the table are in C, thus 5/9 is in C.

(b) To show that 15/37 is in C, try to work backwards: 15/37 could have come from $2 \times 15/37 = 30/37$ or x where $1/(1+x) = 15/37$, ie $1+x = 37/15$, so $x = 37/15 - 1 = 22/15$. But 22/15 is not between 0 and 1, so 22/15 is not in C. So if 15/37 is in C, it must have come from 30/37. Can we show 30/37 is in C? We continue like this. It is easiest to make a table. For each number y in the first columns, y could have come from $2y$ or from x where $1/(1+x) = y$, ie $x = 1/y - 1$.

y	$2y$	$1/y - 1$	$0 < 2y < 1$?	$0 < 1/y - 1 < 1$?
14/15	28/15	14/14 - 1 = 1/14	N	Y
1/14	1/7	14/1 - 1 = 13	Y	N
1/7				
2/7				
4/7				
3/4	...	...	...	...
1/3				
2/3				

Since we are told that 1/2 is in C, tracing back through the table shows that 15/37 is also in C.

3.

4. Note that $2/(x-1) < 0$ is $x < 1$, $2/(x-1) > 0$ if $x > 1$ an $1/(x-2 < 0$ if $x < 2$, $1/(x-2) > 0$ if $x > 2$. Thus if $1 < x < 2$ then $[2/(x-1)] < 0$, yet $[2/(x-1)] > 0$ so there is no solution to the equation for $1 < x < 2$.

   Consider x with $0 \leq x < 1$. Then $-1 \leq x - 1 < 0$, so $2/(x-1) \leq -2$ and hence $[1/(x-1)] \leq -2$. And $-2 \leq x - 2 < -1$, so $-1 < 1/(x-2) \leq -1/2$, and hence $[1/(x-1) = -1$. Thus $[2/(x-1)] \neq [1/(x-2)]$ for $1 \leq x < 1$. No we can assume $x > 2$, so $2/(x-1) > 0$ and $1/(x-2) > 0$. Take integer n with $n > 0$ and we find when $[2/(x-1)] = n$ and $[1/(x-2)] = n$. To have $[2/(x-1)] = n$ means $n \leq 2/(x-1) \leq n+1$ and so $1/n \geq (x-1)/2 > 1/(n+1)$ and thus $1 + 2/(n+1) < x \leq 1 + 2/n$. Similarly, to have $[1/(x-2] = n$ means $n \leq 1/(x-2) < n+1$ and so $1/n \geq x - 2 > 1/(n+1)$ and thus $2 + 1/(n+1) < x \leq 2 + 1/n$. If $n \geq 2$, $1 + 2/n \leq 2 < 2 + 1/(n+1)$ and these two intervals for x do not overlap. So there are no solutions to the equation for these values of n. If $n = 1$, $[2/(x-1] = 1$ for $2 < x \leq 3$ and $[1/(x-2)] = 1$ for $2\frac{1}{2} < x \leq 3$.

   Thus $[2/(x-1)] = [1/(x-2)] = 1$ for $2\frac{1}{2} < x \leq 3$. Thus $[2/(x-1)] = [1/(x-2)] = 1$ for $2\frac{1}{2} < x \leq 3$.

   For $n = 0$, we have $[2/(x-1)] = 0$ when $0 \leq 2/(x-1) < 1$ and so $(x-1)/2 > 1$, ie when $x > 3$. And $[1/(x-2)] = 0$ when $0 \leq 1/(x-2) < 1$ and so $x - 2 > 1$, ie when $x > 3$. Then $[2/(x-1)] = [1/(x-2)] = 0$ for $x > 3$.

   Thus we find that for $x \geq 0$, $[2/(x-1)] = [1/(x-2)]$ just when $x \geq 2\frac{1}{2}$.