Voltage Graphs: An Introduction

**Definition 1.** A graph $G$ consists of a set $V_G$ of vertices and a set $E_G$ of edges. Each edge $e$ has an endpoint set $V_G(e)$ containing either one or two elements of the vertex set $V_G$. The set $I_G = \{V_G(e) \mid e \in E_G\}$ is called the *incidence structure*.

**Definition 2.** A graph is called *simplicial* if it has no self-loops or multiple edges.

**Example 1.** Let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$ where $V(e_1) = \{v_1, v_2\}$, $V(e_2) = \{v_1, v_3\}$, $V(e_3) = \{v_2, v_3\}$ and $V(e_4) = \{v_3, v_4\}$.

![Example 1 Graph]

**Example 2.** We can have different drawings of the same graph.

![Example 2 Graphs]

**Definition 3.** A *direction* for an edge $e$ is an onto function $\{BEGIN, END\} \to V(e)$. The images of $BEGIN$ and $END$ are called the *initial point* and the *end point*.

In topological graph theory we consider each edge $e$ (even loops) to have two directions, arbitrarily distinguished as the *plus direction*, $e^+$, and the *minus direction*, $e^-$. 

![Example 3 Graph]
Definition 4. An edge with a direction is called a **directed edge**.

Definition 5. A **graph map**, \( f : G \rightarrow G' \), consists of a vertex function \( V_G \rightarrow V_{G'} \) and an edge function \( E_G \rightarrow E_{G'} \) such that incidence is preserved.

Under a graph map a loop may be the image of a proper edge, but a proper edge may never be the image of a loop.

Example 3. \( G \)

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\begin{array}{c}
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\text{\includegraphics[width=0.2\textwidth]{example3a}} \quad \text{\includegraphics[width=0.2\textwidth]{example3b}}
\end{array}
\end{array}
\]

\( f : G \rightarrow G' \) such that \( f(v_1) = v'_1, f(v_2) = f(v_3) = v'_2, f(e_1) = f(e_2) = e'_1 \) and \( f(e_3) = e'_2 \).

Definition 6. A graph map \( G \rightarrow G' \) is called an **isomorphism** if both its vertex function and its edge function are bijections.

Example 4.

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\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{example4a}} \quad \text{\includegraphics[width=0.2\textwidth]{example4b}}
\end{array}
\end{array}
\]

Definition 7. An isomorphism from a graph onto itself is called an **automorphism**. Under the operation of composition, the family of all automorphisms of a graph forms a group, called the **automorphism group of the graph** and is denoted \( Aut(G) \).
Definition 8. Let $G$ be a graph, and let $B$ be a group such that for each element $b \in B$, there is a graph automorphism $\phi_b : G \to G$ and such that the following two conditions hold:

(i) If $1$ is the group identity, then $\phi_1 : G \to G$ is the identity automorphism.

(ii) For all $b, c \in B$, $\phi_b \circ \phi_c = \phi_{bc}$.

Then the group $B$ is said to act on (the left of) the graph $G$. If, moreover the additional condition

(iii) For every group element $b \neq 1$, there is no vertex $v \in V_G$ such that $\phi_b(v) = v$ and no edge $e \in E_G$ such that $\phi(e) = e$

holds, then $B$ is said to act without fixed points or act freely on $G$.

Definition 9. For any vertex $v$ of the graph $G$, the orbit $[v]$ (or $[v]_B$) is defined to be the set $\{\phi_b(v) \mid b \in B\}$. Similarly, for any edge $e$, the orbit $[e]$ (or $[e]_B$) is defined to be the set $\{\phi_b(e) \mid b \in B\}$. The set of vertex orbits and edge orbits are denoted $V/B$ and $E/B$ respectively.

Example 5. Consider the Petersen graph $P$. For $i \in \mathbb{Z}_5$, let $\phi_i$ be the rotation $\frac{2\pi i}{5}$ radians. Under the action of $\mathbb{Z}_5$ there are two vertex orbits and three edge orbits.

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{petersen_graph.png}
\end{array}
\]

Definition 10. The regular quotient $G/B$ is the graph with vertex set $V/B$ and edge set $E/B$ such that the vertex orbit $[v]$ is an endpoint if the edge orbit $[e]$ if any vertex in $[v]$ is an endpoint of any edge in $[e]$.


\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{petersen_quotient.png}
\end{array}
\]
Definition 11. Associated with a graph quotient there is a vertex function \( v \rightarrow [v] \) and an edge function \( e \rightarrow [e] \) that together are called the quotient map \( q_B : G \to G/B \).

Definition 12. A graph map \( p : \tilde{G} \to G \) is called a regular covering projection if it is equivalent to a quotient map as follows:

There exists a group \( B \) that acts freely on \( \tilde{G} \), and there exists an isomorphism \( i : \tilde{G}/B \to G \) such that \( i \circ q_B = p \), where \( q_B : \tilde{G} \to \tilde{G}/B \).

The range graph \( G \) is called the base (or base space), and the domain graph \( \tilde{G} \) is called a regular covering (or regular covering space).

Definition 13. Let \( G \) be a graph whose edges have all been given plus and minus directions, and let \( \mathcal{A} \) be a group (generally finite). A set function \( \alpha \) from the + directed edges of \( G \) into the group \( \mathcal{A} \) is called an ordinary voltage assignment on \( G \), and the pair \( (G, \alpha) \) an ordinary voltage graph. The values of \( \alpha \) are called the voltages and \( \mathcal{A} \) is called the voltage group.

The purpose if assigning voltages to the graph \( G \), called the base graph or the base, is to obtain the object called the (right) derived graph, \( G^\alpha \).

The vertex set of \( G^\alpha \) is the cartesian product \( V_G \times \mathcal{A} \) and the edge set of \( G^\alpha \) is the cartesian product \( E_G \times \mathcal{A} \). The vertices of \( G \) are denoted \( v_a \) and the edges of \( G \) are denoted \( e_a \). If the directed edge of \( e^+ \) of the base graph runs from vertex \( u \) to vertex \( v \) and if \( b \) is the voltage assigned to \( e^+ \), then the directed edge \( e^+_a \) of the derived graph \( G^\alpha \) runs from vertex \( u_a \) to vertex \( v_{ab} \).

Example 7. \( \mathcal{A} = \mathbb{Z}_3 \)

\[(G, \alpha) \quad G^\alpha\]
Example 8. $\mathcal{A} = \mathbb{Z}_5$

$\langle G, \alpha \rangle$

\[ G^\alpha \]

Dumbell graph

Definition 14. For every vertex $v$ of $G$, the set of vertices $v_\alpha$ is called the fiber over $v$. Similarly, for every edge $e$ of $G$, the set of edges $e_\alpha$ is called the fiber over $e$.

The fiber over a proper edge is isomorphic to the disjoint union of $|\mathcal{A}|$ copies of $K_2$. The fiber over a loop always forms a set of cycles (if the voltage group is finite).

Example 9. $\mathcal{A} = \mathbb{Z}_6$

$\langle G, \alpha \rangle$

\[ G^\alpha \]

1 has order 6
2 has order 3

The fiber over $d$ forms one 6-cycle. The fiber over $e$ forms two 3-cycles.

In general, if the voltage $b$ on a $v$ based loop $e^+$ has order $n$ in the group $\mathcal{A}$ then each cycle in the edge fiber must have length $n$ and there are $\frac{|\mathcal{A}|}{n}$ such cycles, and these $n$-cycles are mutually disjoint.

Definition 15. The natural projection $p : G^\alpha \to G$ maps every vertex in the fiber over $v$ to $v$ and every edge in the fiber over $e$ to $e$. 

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The voltage on a minus directed edge $e^-$ is the group inverse of the voltage on $e^+$, that is $\alpha(e^-) = (\alpha(e^+))^{-1}$.

**Definition 16.** The net voltage on a walk $W = e_1^{\sigma_1}, e_2^{\sigma_2}, \ldots, e_n^{\sigma_n}$ (each $\sigma_i = +$ or $-$) is defined to be the product $\alpha(e_1^{\sigma_1})\alpha(e_2^{\sigma_2}) \cdots \alpha(e_n^{\sigma_n})$.

**Example 10.** Walk $e^+, d^-, c^-, d^+$

$$\alpha(e^+) + \alpha(d^-) + \alpha(c^-) + \alpha(d^+) = 2 + 0 + 3 + 0 = 1$$

**$\mathbb{Z}_4$**

**Definition 17.** A lift of a walk $W = e_1^{\sigma_1}, e_2^{\sigma_2}, \ldots, e_n^{\sigma_n}$ in the base $G$ is the walk $\bar{W} = \bar{e}_1^{\sigma_1}, \bar{e}_2^{\sigma_2}, \ldots, \bar{e}_n^{\sigma_n}$ in the derived graph $G^\alpha$ such that for $1 \leq i \leq n$ the edge $\bar{e}_i$ is in the fiber over the edge $e_i$.

**Example 11.** $\alpha = \mathbb{Z}_3$

walk $c^-, e^+$ has lifts $c_1^-, e_1^+$ $u_0$ to $v_2$

walk $c^-, e^-$ has lifts $c_2^-, e_2^-$ $u_0$ to $v_1$

walk $c^-, e_1^+$ has lifts $u_2$ to $v_1$
Theorem 1. Let $W$ be a walk in an ordinary voltage graph $(G, \alpha)$ such that the initial vertex of $W$ is $u$. Then for each vertex $u_a$ in the fiber over $u$, there is a unique lift of $W$ that starts at $u_a$.

The lift of the walk $W$ starting at vertex $u_a$ is denoted $W_a$.

Theorem 2. Let $W$ be a walk from $u$ to $v$ in an ordinary voltage graph $(G, \alpha)$, and let $b$ be the net voltage on $W$. Then the lift $W_a$ starting at $u_a$ terminates at the vertex $v_{ab}$.

Theorem 3. Let $C$ be a $k$-cycle on the base space of an ordinary voltage graph $(G, \alpha)$ such that the net voltage on $C$ has order $m$ in the voltage group $\mathcal{A}$. Then each component of the preimage $p^{-1}(C)$ in a $km$-cycle, and there are $\frac{|\mathcal{A}|}{m}$ such components.

Example 12. $\mathcal{A} = \mathbb{Z}_2$

walk $e_1^+, e_2^+, e_3^+, e_4^+$
net voltage $0 \pmod{2}$
order 1

preimage $(e_1,0), (e_2,1), (e_3,0), (e_4,1)$ and $(e_1,1), (e_2,0), (e_3,1), (e_4,0)$
two 4-cycles
$\frac{2}{1}$ 4x1

walk $e_1^+, e_2^+, e_5^+$
net voltage $1 \pmod{2}$
order 2

preimage $(e_1,0), (e_2,1), (e_5,0), (e_1,1), (e_2,0), (e_5,1)$
one 6-cycle
$\frac{2}{2}$ 3x2
Which graphs are derivable with ordinary voltages?

Let $\langle G, \alpha \rangle$ be a voltage graph; $\forall c \in \mathcal{A}$ let $\phi_c : G^\alpha \to G^\alpha$ such that $\phi_c(v_a) = v_{ca}$ and $\phi_c(e_a) = e_{ca}$. Obviously $\phi_1 : G^\alpha \to G^\alpha$ is the identity automorphism and no other $\phi_c$ fixes any vertex or edge and $\phi_c \circ \phi_d = \phi_{cd}$. Thus, $\mathcal{A}$ acts freely on the left of $G^\alpha$ and is called the natural (left) action of the voltage group.

Theorem 4. The vertex orbits of the natural action of the voltage group on an ordinary derived graph are the vertex fibres, and the edge orbits are the edge fibres.

Corollary 5. The natural projection $p : G^\alpha \to G$ of the derived graph onto the base graph is a covering projection.

A free group action on a graph may be identified with a subgroup of $\text{Aut}(G)$ in which every automorphism except the identity is fixed-point free. Every regular quotient arises from such a subgroup, if the only such subgroup is trivial, then the only regular quotient is the graph itself.

References