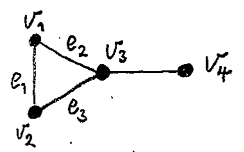
Voltage Graphs: An Introduction

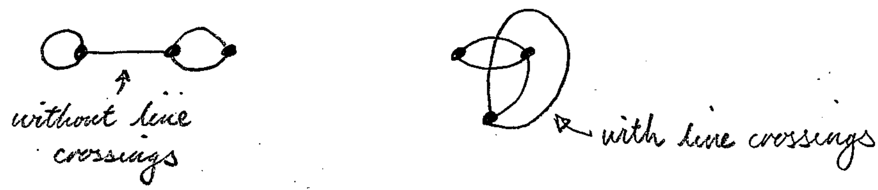
Definition 1. A *graph* G consists of a set V_G of vertices and a set E_G of edges. Each edge e has an endpoint set $V_G(e)$ containing either one or two elements of the vertex set V_G. The set $I_G = \{V_G(e) \mid e \in E_G\}$ is called the *incidence structure*.

Definition 2. A graph is called *simplicial* if it has no self-loops or multiple edges.

Example 1. Let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$ where $V(e_1) = \{v_1, v_2\}$, $V(e_2) = \{v_1, v_3\}$, $V(e_3) = \{v_2, v_3\}$ and $V(e_4) = \{v_3, v_4\}$.

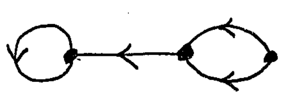


Example 2. We can have different drawings of the same graph.



Definition 3. A *direction* for an edge e is an onto function $\{BEGIN, END\} \to V(e)$. The images of $BEGIN$ and END are called the *initial point* and the *end point*.

In topological graph theory we consider each edge e (even loops) to have two directions, arbitrarily distinguisehed as the *plus direction*, e^+, and the *minus direction*, e^-.

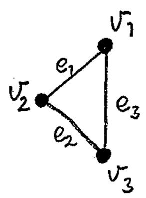


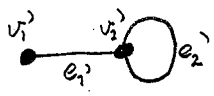
Definition 4. An edge with a direction is called a *directed edge.*

Definition 5. A *graph map*, $f : G \to G'$, consists of a vertex function $V_G \to V_{G'}$ and an edge function $E_G \to E_{G'}$ such that incidence is preserved.

Under a graph map a loop may be the image of a proper edge, but a proper edge may never be the image of a loop.

Example 3. G G'

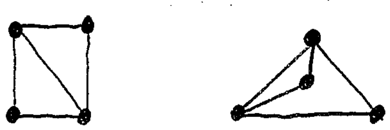




$f : G \to G'$ such that $f(v_1) = v_1'$, $f(v_2) = f(v_3) = v_2'$, $f(e_1) = f(e_2) = e_1'$ and $f(e_3) = e_2'$.

Definition 6. A graph map $G \to G'$ is called an *isomorphism* if both its vertex function and its edge function are bijections.

Example 4.



Definition 7. An isomporphism from a graph onto itself is called an *automorphism.* Under the operation of composition, the family of all automorphisms of a graph forms a group, called the *automorphism group of the graph* and is denoted $Aut(G)$.

Definition 8. Let G be a graph, and let $\mathcal{B}$ be a group such that for each element $b \in \mathcal{B}$, there is a graph automorphism $\phi_b : G \to G$ and such that the following two conditions hold:

(i) If 1 is the group identity, then $\phi_1 : G \to G$ is the identity automorphism.

(ii) For all $b, c \in \mathcal{B}$, $\phi_b \circ \phi_c = \phi_{bc}$.

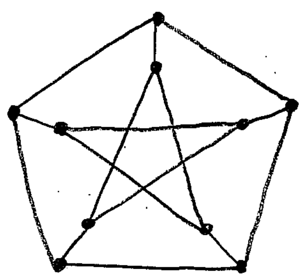
Then the group $\mathcal{B}$ is said to *act on (the left of) the graph G*. If, moreover the additional conidition

(iii) For every group element $b \neq 1$, there is no vertex $v \in V_G$ such that $\phi_b(v) = v$ and no edge $e \in E_G$ such that $\phi(e) = e$

holds, then $\mathcal{B}$ is said to *act without fixed points* or *act freely* on G.

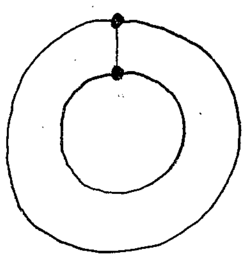
Definition 9. For any vertex v of the graph G, the *orbit* $[v]$ (or $[v]_{\mathcal{B}}$) is defined to be the set $\{\phi_b(v) \mid b \in \mathcal{B}\}$. Similarly, for any edge e, the *orbit* $[e]$ (or $[e]_{\mathcal{B}}$) is defined to be the set $\{\phi_b(e) \mid b \in \mathcal{B}\}$. The set of vertex orbits and edge orbits are denoted $V/\mathcal{B}$ and $E/\mathcal{B}$ respectively.

Example 5. Consider the Petersen graph P. For $i \in \mathbb{Z}_5$, let ϕ_i be the rotation $\frac{2\pi i}{5}$ radians. Under the action of $\mathbb{Z}_5$ there are two vertex orbits and three edge orbits.



Definition 10. The *regular quotient* $G/\mathcal{B}$ is the graph with vertex set $V/\mathcal{B}$ and edge set $E/\mathcal{B}$ such that the vertex orbit $[v]$ is an endpoint if the edge orbit $[e]$ if any vertex in $[v]$ is an endpoint of any edge in $[e]$

Example 6. $P/\mathbb{Z}_5$



Definition 11. Associated with a graph quotient there is a vertex function $v \longrightarrow [v]$ and an edge function $e \longrightarrow [e]$ that together are called the *quotient map* $q_{\mathcal{B}} : G \longrightarrow G/\mathcal{B}$.

Definition 12. A graph map $p : \bar{G} \longrightarrow G$ is called a *regular covering projection* if it is equivalent to a quotient map as follows:

> There exists a group $\mathcal{B}$ that acts freely on $\bar{G}$, and there exists an isomorphism $i : \bar{G}/\mathcal{B} \longrightarrow G$ such that $i \circ q_{\mathcal{B}} = p$, where $q_{\mathcal{B}} : \bar{G} \longrightarrow \bar{G}/\mathcal{B}$.

The range graph G is called the *base* (or *base space*), and the domain graph $\bar{G}$ is called a *regular covering* (or *regular covering space*).

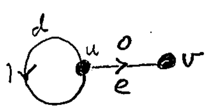
Definition 13. Let G be a graph whose edges have all been given plus and minus directions, and let $\mathcal{A}$ be a group (generaly finite). A set function α from the $+$ directed edges of G into the group $\mathcal{A}$ is called an *ordinary voltage assignment* on G, and the pair $\langle G, \alpha \rangle$ an *ordinary voltage graph.* The values of α are called the *voltages* and $\mathcal{A}$ is called the voltage *group*.

The purpose if assigning voltages to the graph G, called the *base graph* or the *base*, is to obtain the object called the *(right) derived graph*, G^α.

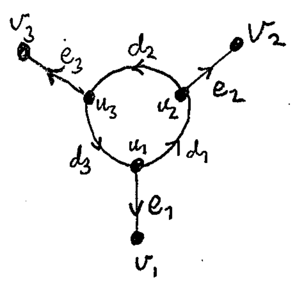
The vertex set of G^α is the cartesian product $V_g \times \mathcal{A}$ and the edge set of G^α is the cartesian product $E_G \times \mathcal{A}$. The vertices of G are denoted v_a and the edges of G are denoted e_a. If the directed edge of e^+ of the base graph runs from vertex u to vertex v and if b is the voltage assigned to e^+, then the directed edge e_a^+ of the derived graph G^α runs from vertex u_a to vertex v_{ab}.

Example 7. $\mathcal{A} = \mathbb{Z}_3$

$\langle G, \alpha \rangle$

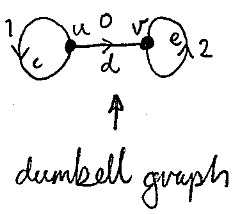


G^α

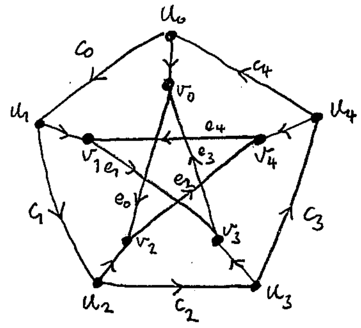


Example 8. $\mathcal{A} = \mathbb{Z}_5$

$\langle G, \alpha \rangle$



G^α

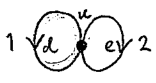


Definition 14. For every vertex v of G, the set of vertices v_a is called the *fiber* over v. Similarly, for every edge e of G, the set of edges e_a is called the *fiber* over e.

The fiber over a proper edge is isomorphic to the disjoint union of $|\mathcal{A}|$ copies of K_2. The fiber over a loop always forms a set of cycles (if the voltage group is finite).

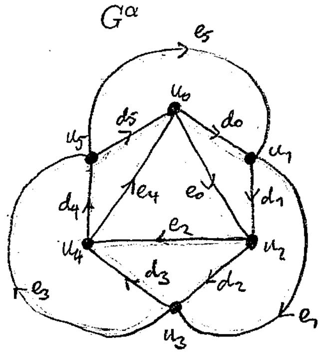
Example 9. $\mathcal{A} = \mathbb{Z}_6$.

$\langle G, \alpha \rangle$



1 has order 6
2 has order 3

G^α



The fiber over d forms one 6-cycle. The fiber over e forms two 3-cycles.

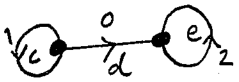
In general, if the voltage b on a v based loop e^+ has order n in the group $\mathcal{A}$ then each cycle in the edge fiber must have length n and there are $\frac{|\mathcal{A}|}{n}$ such cycles, and these n-cycles are mutually disjoint.

Definition 15. The *natural projection* $p : G^\alpha \to G$ maps every vertex in the fiber over v to v and every edge in the fiber over e to e.

The voltage on a minus directed edge e^- is the group inverse of the voltage on e^+, that is $\alpha(e^-) = (\alpha(e^+))^{-1}$.

Definition 16. The *net voltage* on a walk $W = e_1^{\sigma_1}, e_2^{\sigma_2}, \ldots, e_n^{\sigma_n}$ (each $\sigma_i = +$ or $-$) is defined to be the product $\alpha(e_1^{\sigma_1})\alpha(e_2^{\sigma_2})\ldots\alpha(e_n^{\sigma_n})$.

Example 10. Walk e^+, d^-, c^-, d^+



$\alpha(e^+) + \alpha(d^-) + e(c^-) + \alpha(d^+)$

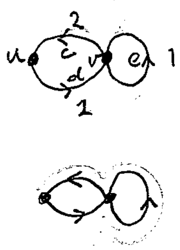
$= 2 + 0 + 3 + 0 = 1$

$\mathbb{Z}_4$

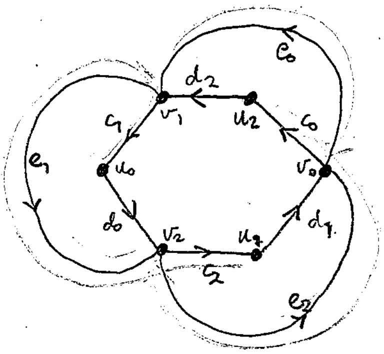
Definition 17. A *lift* of a walk $W = e_1^{\sigma_1}, e_2^{\sigma_2}, \ldots, e_n^{\sigma_n}$ in the base G is the walk $\bar{W} = \bar{e}_1^{\sigma_1}, \bar{e}_2^{\sigma_2}, \ldots, \bar{e}_n^{\sigma_n}$ in the derived graph G^α such that for $1 \leq i \leq n$ the edge $\bar{e}_i$ is in the fiber over the edge e_i.

Example 11. $\mathcal{A} = \mathbb{Z}_3$

$\langle G, \alpha \rangle$



G^α



walk c^-, e^+ has lifts

c_1^-, e_1^+ u_0 to v_2

c_2^-, e_2^+ u_1 to v_0

c_0^-, e_0^+ u_2 to v_1

walk c^-, e^- has lifts

c_1^-, e_0^- u_0 to v_0

c_2^-, e_1^- u_1 to v_1

c_0^-, e_2^- u_2 to v_2

Theorem 1. *Let W be a walk in an ordinary voltage graph $\langle G, \alpha\rangle$ such that the initial vertex of W is u. Then for each vertex u_a in the fiber over u, there is a unique lift of W that starts at u_a.*

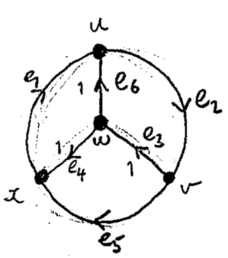
The lift of the walk W starting at vertex u_a is denoted W_a.

Theorem 2. *Let W be a walk from u to v in an ordinary voltage graph $\langle G, \alpha\rangle$, and let b be the net voltage on W. Then the lift W_a starting at u_a terminates at the vertex v_{ab}.*

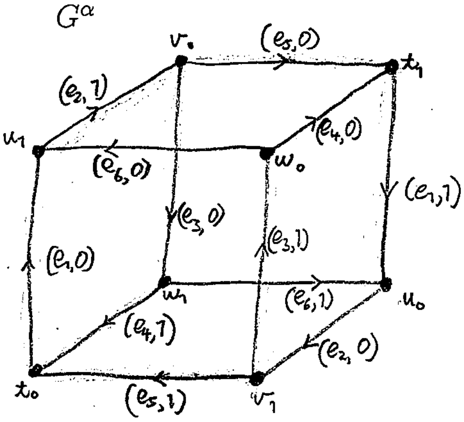
Theorem 3. *Let C be a k-cycle on the base space of an ordinary voltage graph $\langle G, \alpha\rangle$ such that the net voltage on C has order m in the voltage group $\mathcal{A}$. Then each component of the preimage $p^{-1}(C)$ in a km-cycle, and there are $\frac{|\mathcal{A}|}{m}$ such components.*

Example 12. $\mathcal{A} = \mathbb{Z}_2$

$\langle G, \alpha\rangle$



G^α



walk $e_1^+, e_2^+, e_3^+, e_4^+$

net voltage $0 \pmod 2$

order 1

preimage $(e_1,0), (e_2,1), (e_3,0), (e_4,1)$ and

$(e_1,1), (e_2,0), (e_3,1), (e_4,0)$

two 4-cycles

$\frac{2}{1}$ 4×1

walk e_1^+, e_2^+, e_5^+

net voltage $1 \pmod 2$

order 2

preimage $(e_1,0), (e_2,1), (e_5,0), (e_1,1), (e_2,0), (e_5,1)$

one 6-cycle

$\frac{2}{2}$ 3×2

Which graphs are derivable with ordinary voltages?

Let $\langle G, \alpha \rangle$ be a voltage graph; $\forall c \in \mathcal{A}$ let $\phi_c : G^\alpha \longrightarrow G^\alpha$ such that $\phi_c(v_a) = v_{ca}$ and $\phi_c(e_a) = e_{ca}$. Obviously $\phi_1 : G^\alpha \longrightarrow G^\alpha$ is the identity automorphism and no other ϕ_c fixes any vertex or edge and $\phi_c \circ \phi_d = \phi_{cd}$. Thus, $\mathcal{A}$ acts freely on the left of G^α and is called the *natural (left) action* of the voltage group.

Theorem 4. *The vertex orbits of the natural action of the voltage group on an ordianry derived graph are the vertex fibres, and the edge orbits are the edge fibres.*

Corollary 5. *The natural projection $p : G^\alpha \longrightarrow G$ of the derived graph onto the base graph is a covering projection.*

A free group action on a graph may be identified with a subgroup of $Aut(G)$ in which every automorphism except the identity is fixed-point free. Every regular quotient arises from such a subgroup, if the only such subgroup is trivial, then the only regular quotient is the graph itself.

References

[1] J.L. Gross and T.W.Tucker. Topological Graph Theory, Dover Publications Inc., New York (2001)