

**Disclaimer**

This document is in no way guaranteed to be correct, complete or continuous. At the moment, particularly, the document is still in construction, and so not all lemma numbers match up, etc, and the diagrammatic approach to co-algebras has been omitted temporarily.

Any errors should be picked up by the avid readers of this document, and an email sent to Joel Gilmore ([jbg@maths.uq.edu.au](mailto:jbg@maths.uq.edu.au)) or Brett Witty ([bew@maths.uq.edu.au](mailto:bew@maths.uq.edu.au)), and corrections will be made as soon as possible, and a note posted on the website.

Thankyou, and enjoy!



# MATH3103

## 1.1 Vector Spaces and Constructions

By a vector space  $V$  we mean a vector space over the complex field  $\mathbb{C}$ .

If  $W$  is another vector space and  $\varphi : V \rightarrow W$  a vector space (linear) map then:

$$\varphi(V) \equiv \{\varphi(v) \mid v \in V\}$$

is a vector subspace of  $W$ . If  $\{v_i\}$  is a basis for  $V$  then  $\{\varphi(v_i)\}$  spans  $\varphi(V)$  since

$$\varphi\left(\sum_i \alpha_i v_i\right) = \sum_i \alpha_i \varphi(v_i) \quad \alpha_i \in \mathbb{C}$$

**Definition 1.**  $\varphi$  is called:

- I) 1-1  $\iff \varphi(v) = 0 \Rightarrow v = 0$ .
- II) onto  $\iff \varphi(V) = W$ .
- III) a vector space isomorphism if  $\varphi$  is 1-1 and onto.

*NOTE:*

For finite dimensional vector spaces,

- I) Clearly  $\varphi$  is 1-1  $\iff \dim \varphi(V) = \dim V$ .
- II)  $\varphi$  is onto  $\iff \dim \varphi(V) = \dim W$ .
- III) Thus  $\varphi$  is an isomorphism  $\iff \varphi$  is 1-1 and  $\dim V = \dim W$ .

NOTATION: The space of linear maps from  $V$  to  $W$  forms a vector space, denoted  $\ell(V, W)$ . If  $V = W$  we denote  $\ell(V, W)$  by  $\ell(V)$  for short.

**Lemma 3.1.**

$$\dim \ell(V, W) = \dim V \cdot \dim W.$$

*Proof.* Let  $\{v_i\}_{i=1}^n$  be a basis for  $V$  and  $\{w_\alpha\}_{\alpha=1}^m$  a basis for  $W$ . Then a basis for  $\ell(V, W)$  is given by  $f_{i\alpha} \in \ell(V, W)$ , defined by:

$$f_{i\alpha}(v_j) = \delta_{ij}w_\alpha$$

Clearly the  $f_{i\alpha}$  are linearly independent. To show they span  $\ell(V, W)$  let  $f \in \ell(V, W)$  be arbitrary and write:

$$f(v_i) = \sum_a c_{\alpha i} w_\alpha, \quad c_{\alpha i} \in \mathbb{C}$$

Then

$$f = \sum_{i,\alpha} c_{\alpha i} f_{i\alpha}$$

□

## 1.2 Duality

Let  $V$  be a vector space. We call  $\ell(V, \mathbb{C})$  the **dual** of  $V$ , denoted  $V^*$ . From Lemma 1.1,  $\dim V = \dim V^*$ , if  $V$  is finite dimensional. If  $\{v_i\}$  is a basis for  $V$  then we have corresponding dual basis  $\{v_i^*\}$  for  $V^*$  defined by:

$$v_i^*(v_j) = \delta_{ij}$$

Given a linear map  $\varphi : V \rightarrow W$ , we define a **dual map**  $\varphi^* : W^* \rightarrow V^*$  by  $\varphi^*(w^*) \in V^*$  where:

$$\varphi^*(w^*)(v) = w^*(\varphi(v)), \quad w^* \in W^*, v \in V$$

*NOTE:*

We say that "duality reverses arrows". Then  $\varphi \rightarrow \varphi^*$  determines a vector space isomorphism  $\ell(V, W) \rightarrow \ell(W^*, V^*)$ .

## 1.3 Tensor Products

Let  $V, W$  be vector spaces. Then  $V \otimes W$  is the vector space spanned by all products  $v \otimes w$ ,  $v \in V$ ,  $w \in W$  satisfying:

- I)  $v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w$   
 $v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$
- II)  $\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w), \quad \alpha \in \mathbb{C}$

If  $\{v_i\}$  is a basis for  $V$  and  $\{w_\alpha\}$  is a basis for  $W$  then  $v_i \otimes w_\alpha$  is a basis for  $V \otimes W$ , so if  $V, W$  are finite dimensional then

$$\dim V \otimes W = \dim V \cdot \dim W.$$

NOTE:

Axiom II) of the tensor product definition states that

$$0 \otimes v = v \otimes 0 = 0 \in V \otimes W, \quad \forall v \in V, w \in W$$

**Example.**

$$w = \left\{ \begin{pmatrix} p(x) \\ q(x) \end{pmatrix} \mid p, q \in \mathbb{C}[x] \right\} \cong \mathbb{C}^2 \otimes \mathbb{C}[x]$$

where  $\mathbb{C}[x]$  is a vector space with basis  $x^n$ ,  $n = 0, 1, 2, \dots$ ;  $x^0 \cong 1$ .  
Basis for  $W$  is thus

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes x^n \cong \begin{pmatrix} x^n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes x^n \cong \begin{pmatrix} 0 \\ x^n \end{pmatrix}, \quad n = 0, 1, 2, \dots$$

NOTE:  $0 \otimes \mathbb{C}[x] = \mathbb{C}^2 \otimes 0 = (0)$

**Lemma 5.1.** Every  $\xi \in V \otimes W$  is uniquely expressible as:

$$\xi = \sum_i v_i \otimes w_i, \quad \text{for some } w_i \in W.$$

Similarly every  $\eta \in W \otimes V$  is uniquely expressible as:

$$\eta = \sum_i w'_i \otimes v_i, \quad \text{for some } w'_i \in W.$$

*Proof.* If  $\{w_\alpha\}$  is a basis for  $W$  then  $\xi \in V \otimes W$  is uniquely expressible:

$$\xi = \sum_i \sum_\alpha c_{i\alpha} v_i \otimes w_\alpha$$

where  $w_i = \sum_\alpha c_{i\alpha} w_\alpha \in W$ . Similarly for  $W \otimes V$ .

□

If  $C \in \ell(V \otimes W)$ , then:

$$C(v_i \otimes w_\alpha) = \sum_j \sum_\beta c_{j\beta, i\alpha} (v_j \otimes w_\beta)$$

where  $c_{j\beta, i\alpha}$  are the matrix elements of  $C$  in the tensor product basis.

**Example.**

Given  $A \in \ell(V)$ ,  $B \in \ell(W)$ , define  $A \otimes B \in \ell(V \otimes W)$  by:

$$\begin{aligned} (A \otimes B)(v_i \otimes w_\alpha) &= Av_i \otimes Bw_\alpha = \sum_{j,\beta} A_{ji}v_j \otimes B_{\beta\alpha}w_\beta \\ &= \sum_{j,\beta} A_{ji}B_{\beta\alpha}(v_j \otimes w_\beta) \\ \Rightarrow (A \otimes B)_{j\beta,i\alpha} &= A_{ji}B_{\beta\alpha} \end{aligned}$$

**Lemma 6.1.** *We have a vector space isomorphism:*

$$W \otimes V^* \cong \ell(V, W)$$

*Proof.* We define the linear map  $\varphi : W \otimes V^* \longrightarrow \ell(V, W)$ ,  $\varphi(w \otimes v^*) \in \ell(V, W)$  given by:

$$\varphi(w \otimes v^*)(v') = v^*(v')w, \forall v^* \in (V^*), v' \in V, w \in W$$

We first show  $\varphi$  is 1-1. Let  $\{v_i\}$  be a basis for  $V$  with corresponding dual basis  $\{v_i^*\}$  for  $V^*$ . From Lemma 1.3 every  $\xi \in W \otimes V^*$  is expressible:

$$\begin{aligned} \xi &= \sum_i w_i \otimes v_i^*, & w_i \in W \\ \Rightarrow \varphi(\xi)(v) &= \sum_i v_i^*(v)w_i, & \forall v \in V. \end{aligned}$$

Thus

$$\begin{aligned} \varphi(\xi) = 0 \Rightarrow 0 &= \varphi(\xi)(v_j) \\ &= \sum_i \underbrace{v_i^*(v_j)}_{\delta_{ij}} w_i = w_j, & \forall j \end{aligned}$$

$\Rightarrow \xi = 0$ .  $\therefore \varphi$  is 1-1. To show  $\varphi$  is onto note that  $\ell(V, W)$  is spanned by linear maps

$$f_{i\alpha}(v_j) = \delta_{ij}w_\alpha$$

Then it is easily seen that

$$\varphi(w_\alpha \otimes v_i^*) = f_{i\alpha}$$

□

## 1.4 Bi-linear Forms

$\varphi \in \ell(V \otimes V, \mathbb{C}) = (V \otimes V)^*$  is called a **bilinear form** on  $V$ .

Following usual notation we set

$$(v, w) = \varphi(v \otimes w), \quad \forall v, w \in V.$$

**Definition 2.**  $(, )$  is called **non-degenerate** if

$$(v, w) = 0, \forall w \in V \Rightarrow v = 0$$

and

$$(w, v) = 0, \forall w \in V \Rightarrow v = 0$$

We call  $(, )$  **symmetric** if

$$(v, w) = (w, v), \forall v, w \in V.$$

## 1.5 Matrix of $(, )$ and dual basis for $V$ (finite dimensional)

Let  $\{v_i\}_{i=1}^n$  be a basis for  $V$ . We call matrix  $g$ , with entries

$$g_{ij} = (v_i, v_j)$$

the **matrix** of the bilinear form.

**Exercise.** I) Show that  $(, )$  is non-degenerate  $\iff g$  is a non-singular matrix. Moreover  $(, )$  is **symmetric**  $\iff g$  is a symmetric matrix.

II) If  $V, W$  are finite dimension vector spaces, prove that

$$V \otimes W^* = V^* \otimes W^*$$

## 1.6 Matrix Elements and non-degenerate bilinear forms

Let  $(, )$  be a **non-degenerate** bilinear form on  $V$  (finite dimensional) and  $\{v_i\}_{i=1}^n$  a basis for  $V$ . We define a **left** dual basis with respect to  $(, )$  by

$$(v^i, v_j) = \delta_j^i$$

**Exercise.**

Set  $g_{ij} = (v_i, v_j)$ ,  $g^{ij} = (g^{-1})_{ij}$ . Show that  $v^i = g^{ij} v_j$  (summation over  $j$ )

Given  $v = \sum_{j=1}^n \alpha_j v_j \in V, \alpha_j \in \mathbb{C}$

$$\Rightarrow (v^i, v) = \sum_{j=1}^n \alpha_j \underbrace{(v^i, v_j)}_{\delta_{ij}} = \alpha_i$$

$$\therefore v = \sum_{i=1}^n (v^i, v) v_i$$

In particular, given  $A \in \ell(V)$ ,

$$Av_i = \sum_{j=1}^n (v^j, Av_i) v_j$$

$\Rightarrow A_{ji} = (v^j, Av_i)$  are the matrix elements of  $A$  in the basis  $\{v_i\}_{i=1}^n$

## 1.7 Conventions and Other Constructions

1. Given vector space  $V$ ,  $I_V$  (or just  $I$ )  $\in \ell(V)$  denotes the **identity map** on  $V$  defined by:

$$I(v) = v, \quad \forall v \in V$$

2. Given vector spaces  $V, W$ ,  $T : V \otimes W \rightarrow W \otimes V$  denotes the **twist map** defined by:

$$T(v \otimes w) = w \otimes v, \quad \forall v \in V, w \in W.$$

3. May also consider tensor products of 3 or more vector spaces. eg.  $U, V, W$  are vector spaces, we may define:

$$U \otimes V \otimes W = (U \otimes V) \otimes W = U \otimes V \otimes W.$$

etc.

### 1.7.1 Direct Sums

1. Given vector spaces  $V, W$  we may define vector space

$$V + W = \{v + w | v \in V, w \in W\} = W + V$$

Note that  $V, W \subseteq V + W$ . If  $V \cap W = (0)$  we write this as

$$V \oplus W \text{ - direct sum (denoting no vectors in common)}$$

If  $B_v = \{v_i\}$  is a basis for  $V$  and  $B_w = \{w_\alpha\}$  is a basis for  $W$ , then  $B_v \cup B_w$  is a basis for  $V \oplus W$ , giving  $\dim V \oplus W = \dim V + \dim W$

*NOTE:*

In general, for  $V, W$  finite dimensional,

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$$

2. Given 3 vector space  $U, V, W$ , we may similarly define

$$U + V + W = \{u + v + w | u \in U, v \in V, w \in W\}$$

If  $U \cap (V + W) = V \cap W = 0$  we write this as

$$U \oplus V \oplus W$$



## Chapter 2

# Algebras

An algebra is a vector space  $A$  equipped with a vector space map

$$m : A \otimes A \rightarrow A \quad \text{- product map}$$

For  $a, b \in A$  we usually write

$$m(a \otimes b) = ab \in A$$

Zero element: Since in  $A \otimes A$ ,  $0 \otimes a = a \otimes 0 = 0$  and  $m$  is linear,

$$m(0 \otimes a) = m(a \otimes 0) = 0$$

or

$$0 \cdot a = a \cdot 0 = 0 \quad \forall a \in A$$

### 2.1 Commutative algebras

An algebra  $A$  is called **commutative** iff  $ab = ba \quad \forall a, b \in A$

In terms of  $m$  this is equivalent to

$$\begin{aligned} m(a \otimes b) = m(b \otimes a) &= mT(a \otimes b) \\ \Rightarrow m &= mT \end{aligned}$$

### 2.2 Associative Algebras

$A$  is called **associative** iff  $a(bc) = (ab)c, \quad \forall a, b, c \in A$

In terms of  $m$  this is equivalent to

$$\begin{aligned} m(a \otimes m(b \otimes c)) &= m(m(a \otimes b) \otimes c) \\ \Rightarrow (I \otimes m)(a \otimes b \otimes c) &= m(m \otimes I)(a \otimes b \otimes c) \\ \Rightarrow m(I \otimes m) &= m(m \otimes I) \end{aligned}$$

## 2.3 Identity Element

An element  $e \in A$  satisfying

$$ea = ae = a \quad \forall a \in A$$

is called the identity element (unique if it exists)

*NOTE:*

Last axiom implies we have a linear map  $u : \mathbb{C} \rightarrow A$ ,  $u(1) = e$  such that

$$m(a \otimes u(1)) = m(u(1) \otimes a) = a, \quad \forall a \in A$$

$u$  is called the unit of  $A$ . We often denote the identity element by  $1 \in A$

**Example.** 1.  $\mathbb{C}$ ,  $\mathbb{C}[x]$  are both associative algebras under normal multiplication

2. Space  $M_n$  of  $m \times n$  complex matrices is an associative algebra under normal matrix product.

3.  $M_n$  is also an algebra with the product  $m(A \otimes B) = [A, B]$  - the commutator

4. Let  $G$  be a finite group and  $\bar{G}$  the vector space spanned by  $G$ . Thus elements of  $\bar{G}$  are of form

$$\sum_{g \in G} \alpha_g g, \quad \alpha_g \in \mathbb{C}$$

Then  $\bar{G}$  becomes an algebra in the obvious way with the multiplication induced by that on  $G$  i.e.

$$\left( \sum_{g \in G} \alpha_g g \right) \cdot \left( \sum_{h \in G} \beta_h h \right) = \sum_{g, h \in G} \alpha_g \beta_h gh$$

*NOTE:*

(1),(3) and (4) are associative algebras with identity

*NOTE:*

(1) is commutative but (2) and generally (4) are not commutative.

*NOTE:*

(4) is non-commutative, non-associative and does not have an identity element. It is, in fact, an example of a *Lie algebra* - called  $gl(n, \mathbb{C})$

**Example.**

Given any vector space  $V$ ,  $\ell(V)$  is an associative algebra under composition defined by

$$(f \circ g)(v) = f(g(v)), \quad \forall f, g \in \ell(V), v \in V$$

$\ell(V)$  admits  $I_V$  as an identity element

**Example.**

If  $A, B$  are algebras, so too is  $A \otimes B$  under natural multiplication

**2.4 Ideals**

Consider  $m : A \otimes A \rightarrow A$

A subspace  $I$  of an algebra  $A$  is called **left ideal** if  $A \cdot I \subseteq I$  i.e.  $ab \in I, \forall a \in A, b \in I$

Similarly,  $I$  is called a **right ideal** if  $I \cdot A \subseteq I$  If  $I \subseteq A$  is both a left and right ideal, we call  $I$  a **two-sided ideal**

**Example.**

If  $A$  is commutative, all ideals are two sided

**Example.**

Let  $M_A$  be the algebra with product

$$m(A \otimes B) = [A, B] = -[B, A], \quad \forall A, B \in M_n$$

Then all ideals of  $M_A$  are two-sided - not generally true for non-commutative algebras.

**2.5 Factor Algebras**

Recall that if  $W$  is a subspace of a vector space  $V$ , we may define the **factor space**

$$V/W \equiv \bar{V} = \bar{v} | v \in V, \quad \bar{v} = v + W$$

which becomes a vector space with definition

$$\alpha \bar{v} = \overline{\alpha v}, \quad \bar{v} + \bar{w} = \overline{v + w}, \quad \forall v, w \in V, \alpha \in \mathbb{C}$$

with zero vector  $\bar{0} = W$

NOTE: For  $v \in V$   $\bar{v} = \bar{0}$  iff  $v \in W$

NOTE:

If  $V$  is f.d.,  $\dim V/W = \dim V - \dim W$ , called the co-dimension of  $W$  in  $V$ .

In particular, if  $I$  is a two sided ideal in an algebra  $A$ , we may define the factor space  $\bar{A} = A/I$  which is also an algebra with product

$$m(\bar{a} \otimes \bar{b}) = \overline{ab}, \quad \bar{a} = a + I$$

which is easily seen to be a linear map, provided  $m$  is well defined (i.e.  $m$  maps  $0 \rightarrow 0$ )

Now zero elements of  $\bar{A} \otimes \bar{A}$  are

$$\bar{a} \otimes \bar{b}, \bar{b} \otimes \bar{a}, \quad b \in I, a \in A$$

and their linear combinations. Since  $I$  is a two sided ideal,

$$ab, ba \in I, \quad \forall b \in I, a \in A$$

Therefore,

$$m(\bar{a} \otimes \bar{b}) = \overline{ab} = \bar{0} = \overline{ba} = m(\bar{b} \otimes \bar{a})$$

$\Rightarrow m$  is well defined, and thus a linear map. Thus  $\bar{A} = A/I$  is an algebra - called the **factor algebra** of  $A$  modulus  $I$ .

*NOTE:*

In general if  $I$  is a left or right ideal, the factor space  $A/I$  is not generally an algebra.

**Example.**

Let

$$T_n = \{A \in M_n \mid \text{tr}(A) = 0\} \subseteq M_n$$

Recall that  $M_n$  is a (non-associative) algebra with product

$$m(A \otimes B) = [A, B]$$

Note that  $\dim M_n = n^2$ ,  $\dim T_n = n^2$  and that  $M_n$  decomposes into a vector space direct sum

$$M_n = T_n \oplus \mathbb{C}I, \quad I = \text{identity matrix} \quad (*)$$

Then both  $\mathbb{C}I, T_n$  are two-sided ideals in  $M_n$  - we call  $(*)$  a decomposition into a direct sum of two sided ideals.

*NOTE:*

$T_n$  also forms a subalgebra of  $M_n$  and gives rise to a Lie Algebra - called the special linear algebra, denoted  $sl(n, \mathbb{C})$

## 2.6 Representation of Associative Algebras

Let  $A, B$  be algebras. Call a linear map  $\varphi : A \rightarrow B$  an **algebraic homomorphism** if

$$\varphi(ab) = \varphi(a)\varphi(b), \quad \forall a, b \in A$$

i.e.  $\varphi$  preserves the algebra structure. If  $\varphi$  is 1-1 (resp. onto) we call  $\varphi$  an algebra **monomorphism** (resp. **epimorphism**). If  $\varphi$  is 1-1 and onto, call  $\varphi$  an algebra **isomorphism** - in that case, say the algebras  $A, B$  are isomorphic, and write  $A \cong B$

**Definition 3.** 1. A representation (rep.)  $\pi$  of  $A$  on a vector space  $V$  is an algebra homomorphism  $\pi : A \longrightarrow \ell(V)$

2.  $V$  is called a (left)  $A$ -module if there exists a linear map

$$\mu_v : A \otimes V \longrightarrow V$$

satisfying

$$\mu_v(1 \otimes \mu_v)(a \otimes b \otimes v) = \mu_v(ab \otimes v), \quad \forall a, b \in A, v \in V$$

NOTATION: For simplicity usually write

$$\mu_v(a \otimes v) = av, \quad \forall a \in A, v \in V$$

So condition for an  $A$ -module becomes

$$(ab)v = a(bv)$$

Definitions (1) and (2) are equivalent. If  $\pi : A \longrightarrow \ell(V)$  is a rep of  $A$  on  $V$ , then  $V$  is a (left)  $A$ -module with definition

$$\mu_v(a \otimes v) = \pi(a)v$$

Conversely, if  $V$  is an  $A$ -module, then  $\pi : A \longrightarrow \ell(V)$  defined by

$$\pi(a)v = \mu_v(a \otimes v)$$

defines a rep of  $A$  on  $V$

**Convention:** refer to  $V$  as an  $A$ -module and  $\pi : A \longrightarrow \ell(V)$  as the representation of  $A$  afforded by  $V$ . We call  $\dim V$  the **dimension** of the representation.

**Definition 4.** If  $W, V$  are  $A$ -modules such that  $W \subseteq V$ , we call  $W$  a sub-module of  $V$ .

**Lemma 15.1.** Suppose  $V$  is an  $A$ -module and  $W \subseteq V$  a subspace of  $V$ , then  $W$  is a submodule iff  $AW \subseteq W$  - i.e.  $W$  is stable (or closed) under action of  $A$ .

*Proof.* If  $W$  is closed under the action of  $A$  it must determine an  $A$ -module since the module axiom

$$(ab)w = a(bw) \quad \forall a, b \in A, w \in W \subseteq V$$

automatically holds because it holds on all of  $V$ . □

**Example.** 1. Let  $W$  be a submodule of an  $A$ -module  $V$ . Then the factor space

$$V/W = \bar{v} | v \in V, \quad \bar{v} = v + W$$

determines an  $A$ -module with definition

$$a\bar{v} = \overline{av}, \quad \forall a \in A, \bar{v} \in V/W$$

2. If  $V \neq (0)$  is an  $A$ -module, then  $(0), V$  are submodules of  $V$ . If these are the only submodules then say that  $V$  is an **irreducible module**. The corresponding rep. is called **irreducible**, or an **irrep**.
3. Recall  $M_n$  is an associative algebra under matrix multiplication. If  $V$  is any  $n$ -dim vector space, there exists a natural algebra isomorphism  $\pi : M_n \rightarrow \ell(V)$ . In particular,  $\pi$  determines a rep of  $M_n$  on  $V$  - in fact an irrep.
4. If  $A$  is an associative algebra, then  $A$  itself becomes an  $A$ -module under left-multiplication and corresponding representation is called the **left regular rep**. In this case, the submodules of  $A$  are the left ideals of  $A$ .

**Definition 5.** Let  $V, W$  be  $A$ -modules. by an  $A$ -module homomorphism we mean a linear map  $\varphi : V \rightarrow W$  such that

$$\varphi(av) = a\varphi(v), \quad \forall v \in V, a \in A$$

Similarly, may define  $A$ -module monomorphism (1-1), epimorphism (onto) and isomorphism. If  $\varphi$  is an isomorphism, say that modules  $V, W$  are **isomorphic** and write  $V \cong W$

*NOTE:*

We do not distinguish isomorphic modules.

**Lemma 16.1.** Let  $\varphi : V \rightarrow W$  be an  $A$ -module homomorphism. Then

- (i)  $\ker \varphi = \{v \in V | \varphi(v) = 0\}$  is a submodule of  $V$
- (ii)  $\text{Im } \varphi = \varphi(V)$  is a submodule of  $W$
- (iii) We have an  $A$ -module isomorphism

$$V/\ker \varphi \cong \varphi(V)$$

**Lemma 16.2 (Schur).** Let  $V$  be a f.d. irreducible  $A$ -module and suppose  $T \in \ell(V)$  satisfies

$$aT = Ta$$

That is,

$$aT(v) = T(av)$$

Then

$$T = \alpha I_v, \quad \alpha \in \mathbb{C}$$

*Proof.* It is an elementary result from linear algebra that  $T$  has at least one e-vector  $v \in V$

$$Tv = \alpha v, \quad \alpha \in \mathbb{C}$$

*NOTE:* This is only true for f.d. vector spaces over complex fields.

Now set

$$W = \{w \in V \mid Tw = \alpha w\} \subseteq V$$

and observe that  $W \neq (0)$  since  $v \in W$ . Then  $W$  is stable under action of  $A$  since  $\forall a \in A, w \in W$ ,

$$T(aw) = aTw = \alpha aw \Rightarrow aw \in W$$

Therefore,  $W \neq 0$  is a submodule of  $V$ . since  $V$  is irreducible,  $V = W$  which proves the result.  $\square$

Schur's Lemma has many applications. an important role in physical applications for providing quantum numbers, etc, is played by **central elements**.

### 2.6.1 Central Elements

$z \in A$  is called **central** if

$$za = az, \quad \forall a \in A$$

We denote the set of all central elements by  $Z$  called the **centre** of  $A$ . This forms a commutative sub-algebra of  $A$ .

**Lemma 17.1.** *Let  $V$  be a f.d. irred.  $A$ -module, and  $z \in Z$  a central element. Then  $z$  reduces to a scalar multiple of identity on  $V$ .*

*Thus, by Schur's lemma,  $\exists \alpha \in \mathbb{C}$  s.t.  $zv = \alpha v, \quad \forall v \in V$*

**Exercise.** 1. *Prove that all irred. f.d.  $A$ -modules for a commutative algebra  $A$  are one-dimensional.*

2. *Show that every f.d.  $A$ -module  $V \neq 0$  must contain an irreducible submodule.*

3. *Let  $V \neq 0$  a f.d irreducible  $A$ -module and  $1 \in A$  the identity element. If  $\pi : A \rightarrow \ell(V)$  is the irrep. afforded by  $V$ , show that  $\pi(1) = I_v$*

### 2.6.2 Complete Reducibility

An  $A$ -module  $V$  is called **completely reducible** if it admits a vector space direct sum into irreducible submodules

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_i, \quad V_i \text{ irreducible}$$

**Exercise.**

*Let  $V$  be a f.d  $A$ -module s.t. for every submodule  $W \subseteq V, \exists$  a submodule  $\overline{W} \subseteq V$  s.t.*

$$V = W \oplus \overline{W}$$

*Then  $V$  is completely reducible. (Hint: Use induction together with previous Exercise(2))*

## 2.7 Co-algebras in general

Thys a vector space  $C$  is called a **co-algebra** if it admits a linear map

$$\Delta : C \longrightarrow C \otimes C$$

$C$  is called **co-commutative** if

$$\Delta = T\Delta$$

where  $t : C \otimes C \longrightarrow C \otimes C$ ,  $T(b \otimes c) = c \otimes b$  is the twist map.

$C$  is co-associative iff

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$$

while  $\epsilon : C \longrightarrow \mathbb{C}$  is called a co-unit if

$$(\epsilon \otimes I)\Delta(c) = 1 \otimes c, \quad \forall c \in C$$

$$(I \otimes \epsilon)\Delta(c) = c \otimes 1, \quad \forall c \in C$$

NOTE: Co-unit  $\epsilon$ , if it exists, can be shown to be *unique*

## 2.8 Sweedler's Convention

Following Sweedler, write

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

With this convention, the co-unit property is expressible as

$$\begin{aligned} (\epsilon \otimes I)\Delta(c) &= \sum_{(c)} \epsilon(c_{(1)}) \otimes c_{(2)} & (2.1) \\ &= \sum_{(c)} 1 \otimes \epsilon(c_{(1)})c_{(2)} \\ \Rightarrow \sum_{(c)} 1 \otimes \epsilon(c_{(1)})c_{(2)} &= 0 \end{aligned}$$

Similarly,

$$\sum_{(c)} 1 \otimes c_{(1)}\epsilon(c_{(2)})$$

*NOTE:* 1. Using the diagrammatic approach, may introduce co-ideals and if  $C$  is co-associative, may talk about co-algebra homomorphisms, co-modules, etc – but won't do this here.

2. The concept of a co-algebra, as we have seen, is naturally dual to that of an algebra, but the dual of an algebra is not generally a co-algebra unless  $A$  is f.d.

**Example.** 1.  $M_n$  is the  $n^2$ -dim vector space with basis consisting of all matrices  $E_j^i$ , i.e. 1 in the  $(i, j)$ th position, and zeroes elsewhere.

Then  $M_n$  becomes a co-algebra under co-product

$$\Delta(E_j^i) = E_k^i \otimes E_j^k, \quad \text{summative over } k$$

Clearly,  $\Delta$  is co-associative, since

$$\begin{aligned} (I \otimes \Delta)\Delta(E_j^i) &= E_k^i \otimes \Delta(E_j^k) \\ &= E_k^i \otimes E_l^k \otimes E_j^l \\ &= \Delta(E_j^i) \otimes E_j^l \\ &= (\Delta \otimes I)\Delta(E_j^i) \end{aligned} \tag{2.2}$$

Moreover for co-unit must have

$$E_j^i = \epsilon(E_k^i)E_j^k = E_k^i\epsilon(E_j^k) \Rightarrow \epsilon(E_j^i) = \delta^i$$

So have a co-algebra with co-unit which is co-associative.

2. Let  $\overline{G}$  be a group algebra of a finite group  $G$ . Then  $\overline{G}^*$  has a basis of elements  $g^*$ ,  $g \in G$ , defined by

$$g^*(h) = \delta(g, h) = \{1, \quad g = h; 0, \quad g \neq h\}$$

Then  $\overline{G}^*$  must determine a co-associative co-algebra with co-unit.

**Exercise.**

Determine the co-algebra structure on  $\overline{G}^*$ .

### 2.8.1 Bi-algebras

Suppose  $H$  is an associative algebra with identity  $1 \in H$  and at the same time a co-associative co-algebra with co-product  $\Delta$ . Then we call  $H$  a **bi-algebra** provided  $\Delta : H \rightarrow H \otimes H$  is an algebra homomorphism, i.e.

$$\Delta(ab) = \Delta(a)\Delta(b)$$

If moreover  $H$  has a co-unit  $\epsilon : H \rightarrow \mathbb{C}$  we call  $H$  a bi-algebra with co-unit provided  $\epsilon$  determines an algebra homomorphism, i.e.

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \quad \forall a, b \in H$$

Throughout, we assume all bi-algebras  $H$  admit a co-unit.

*NOTE:* 1. If  $H$  is a bi-algebra with co-unit  $\epsilon$ , then have alg. homomorphism  $\epsilon : H \rightarrow \mathbb{C}$ . So  $\epsilon$  determines a 1-dim rep. of  $H$  on  $\mathbb{C}$  called the identity representation.

2. In notation of Sweedler, algebra homomorphism requirement on  $\Delta$  is equivalent to

$$\begin{aligned} \sum_{(ab)} (ab)_{(1)} \otimes (ab)_{(2)} &= \Delta(ab) \\ &= \Delta(a)\Delta(b) \\ &= \left( \sum_{(a)} a_{(1)} \otimes a_{(2)} \right) \left( \sum_{(b)} b_{(1)} \otimes b_{(2)} \right) \\ &= \sum_{(a),(b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} \end{aligned}$$

**Theorem 20.1.** *Let  $V, W$  be  $H$ -modules over a bi-algebra  $H$ . Then  $V \otimes W$  is also an  $H$ -module under the action*

$$\begin{aligned} a \cdot (v \otimes w) &= \Delta(a)(v \otimes w) \\ &= \sum_{(a)} a_{(1)} v \otimes a_{(2)} w \end{aligned}$$

*Proof.* It suffices to prove module requirement  $(ab) \circ (v \otimes w) = a \circ (b \circ (v \otimes w))$ . To this end, we have

$$\begin{aligned} (ab) \circ (v \otimes w) &= \Delta(ab)(v \otimes w) \\ &= \sum_{(a),(b)} [a_{(1)} b_{(1)}] v \otimes [a_{(2)} b_{(2)}] w \\ &= \sum_{(a),(b)} a_{(1)} (b_{(1)} v) \otimes a_{(2)} (b_{(2)} w) \\ &= \Delta(a) \cdot \left( \sum_{(b)} b_{(1)} v \otimes b_{(2)} w \right) \\ &= \Delta(a) \cdot (\Delta(b)(v \otimes w)) \\ &= a \circ (b \circ (v \otimes w)) \end{aligned}$$

□

*NOTE:* 1. For general algebras, tensor product of two modules is not generally a module.

2. May recursively define tensor product of 3 or more  $H$ -modules which again gives another  $H$ -module.

e.g. Given another  $H$ -module  $U$  may define  $H$ -module

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) = U \otimes V \otimes W$$

- co-associativity implies 1st two modules are homomorphic, so we may remove brackets.

3. Tensor products are fundamental in physics for coupling of physical systems. For this reason the associative algebras of interest in physics are bi-algebras.
4. If  $V, W$  are  $H$ -modules then  $V \otimes W, W \otimes V$  both determine  $H$ -modules. These are **not** isomorphic unless  $H$  is (almost) co-commutative.

## 2.9 Opposite structure

If  $H$  is a bi-algebra with co-product  $\Delta$  and co-unit  $\epsilon$ , then it is also a bi-algebra with the same co-unit  $\epsilon$ , with co-product  $\Delta^T = T\Delta$  - **opposite co-product**

If  $\Delta = \Delta^T$ , call  $H$  co-commutative.

**Example.** 1. **Finite Group Algebra**

$\overline{G}$  admits a co-product defined by

$$\Delta(g) = g \otimes g, \quad \forall g \in G$$

$$\begin{aligned} \Rightarrow \Delta \left( \sum_{g \in G} \alpha_g g \right) &= \sum_{g \in G} \alpha_g \Delta(g) \\ &= \sum_{g \in G} \alpha_g (g \otimes g) \end{aligned}$$

(2.3)

$\Delta$  clearly determines an algebra homomorphism since

$$\begin{aligned} \Delta(gh) &= gh \otimes gh \\ &= (g \otimes g) \cdot (h \otimes h) \\ &= \Delta(g)\Delta(h) \end{aligned}$$

and is co-associative since

$$\begin{aligned} (\Delta \otimes 1)\Delta(g) &= \Delta(g) \otimes g \\ &= g \otimes g \otimes g \\ &= g \otimes \Delta(g) \\ &= (1 \otimes \Delta)\Delta(g), \quad \forall g \in G \end{aligned}$$

To see  $\overline{G}$  admits a co-unit  $\epsilon$ , co-unit property is equivalent to

$$\begin{aligned} g &= \epsilon(g)g \\ &= g\epsilon(g) \\ \Rightarrow \epsilon(g) &= 1, \quad \forall g \in G \end{aligned} \tag{2.4}$$

2. With  $\overline{G}$  as above,  $\overline{G}^*$  is also a bi-algebra. In this case,  $\overline{G}^*$  is not co-commutative unless  $G$  is an abelian group.

*NOTE:*

If  $H$  is any f.d. bi-algebra so too is  $H^*$  with naturally inherited product  $m^0$ , co-product  $\Delta^0$ , unit  $u^0$  and co-unit  $\epsilon^0$  defined by

$$m^0 = \Delta^* \quad \Delta^0 = m^* \quad u^0 = \epsilon^* \quad \epsilon^0 = u^*$$

Then  $H$  is commutative (resp. co-commutative) iff  $H^*$  is co-commutative (resp. commutative)

## 2.10 Convolution product

Let  $H$  be a bi-algebra. Then  $\ell(H)$  becomes an associative algebra under the **convolution product** defined by

$$\begin{aligned} (f * g)(a) &= m \circ (f \otimes g) \Delta(a) \\ &= \sum_{(a)} f(a_{(1)})g(a_{(2)}), \quad \forall a \in H \end{aligned}$$

The identity  $e$  under this product is given by  $e(a) = \epsilon(1) \cdot 1$ , where 1 is the identity of  $H$ .

Indeed,

$$\begin{aligned} (f * e)(a) &= \sum_{(a)} f(a_{(1)})e(a_{(2)}) \\ &= \sum_{(a)} \epsilon(a_{(2)})f(a_{(1)}) \\ &= f \left( \sum_{(a)} \epsilon(a_{(1)})a_{(2)} \right) \\ &= \sum_{(a)} \epsilon(a_{(2)})f(a_{(1)}) \\ &= f(a) \end{aligned}$$

where we have used the co-unit property

$$\sum_{(a)} \epsilon(a_{(1)})a_{(2)} = \sum_{(a)} a_{(1)}\epsilon(a_{(2)})$$

## 2.11 Hopf Algebras (HA's)

Let  $H$  be a bi-algebra. If the identity map  $I$  on  $H$  admits an inverse  $S$  under the convolution product, we call  $H$  a **Hopf algebra** and  $S$  the antipode of  $H$ .

The antipode  $S$  is thus unique and is defined by

$$m \circ (I \otimes S)\Delta(a) = m \circ (S \otimes I)\Delta(a) = \epsilon(a) \cdot 1, \quad \forall a \in H$$

Equivalently, in Sweedler's notation, where

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

the antipode property is expressible as

$$\sum_{(a)} a_{(1)} S(a_{(1)}) = \sum_{(a)} S(a_{(1)}) a_{(2)} = \epsilon(a) \cdot 1, \quad \forall a \in H$$

The antipode plays a crucial role in the theory of HA's. Some of the properties of the antipode  $S$  are summarised below.

**Proposition 1.** *The antipode  $S$  satisfies*

$$(i) \quad \Delta(S(a)) = (S \otimes S)\Delta^T(a), \quad \forall a \in H$$

$$(ii) \quad S(ab) = S(b)S(a)$$

*so that  $S$  determines an algebra anti-homomorphism on  $H$ .*

*Proof.* See Sweedler □

**Corollary 23.1.** (i)  $S(1) = 1$

$$(ii) \quad \epsilon(S(a)) = \epsilon(a), \quad \forall a \in H$$

*Proof.* Exercise for the reader □

## 2.12 Module Constructions

If  $V, W$  are  $H$ -modules,  $\ell(V, W)$  is also an  $H$ -module under action

$$(a \circ f) = \sum_{(a)} a_{(1)} f S(a_{(2)}), \quad \forall a \in H, f \in \ell(V, w)$$

i.e.

$$(a \circ f)(v) = \sum_{(a)} a_{(1)} f(S(a_{(2)})v), \quad \forall v \in V$$

*Proof.*  $\forall a, b \in H, f \in \ell(V, W), v \in V$  we have

$$\begin{aligned}
((ab) \circ f)v &= \sum_{(ab)} (ab)_{(1)} f(S[(ab)_{(2)}]v) \\
&= \sum_{(a)(b)} a_{(1)} b_{(1)} f(S(a_{(2)} b_{(2)})v) \\
&= \sum_{(a)(b)} a_{(1)} b_{(1)} f(S(a_{(2)})S(a_{(2)})v) \\
&= \sum_{(a)} a_{(1)} (b \circ f)(S(a_{(2)})v) \\
&= a \circ (b \circ f)(v) \\
\Rightarrow (ab) \circ f &= a \circ (b \circ f)
\end{aligned}$$

□

### 2.13 Dual Module

In particular, setting  $W = \mathbb{C}$  which is trivially an  $H$ -module under the action determined by the co-unit  $\epsilon$ , it follows that

$$V^* = \ell(V, \mathbb{C})$$

is also an  $H$ -module under action

$$\begin{aligned}
(a \circ f)(v) &= \sum_{(a)} a_{(1)} f(S(a_{(2)})v) \\
&= \sum_{(a)} \epsilon(a_{(1)}) f(S(a_{(2)})v) \\
&= f[(S(\sum_{(a)} \epsilon(a_{(1)})a_{(2)})v)] \\
&= f(S(a)v), \quad \forall a \in H, f \in V^*, v \in V
\end{aligned}$$

**Proposition 2.** *Let  $V, W$  be  $H$ -modules.*

*Then we have an  $H$ -module isomorphism*

$$W \otimes V^* \cong \ell(V, W)$$

*Proof.* From Lemma (3), there is a natural vector space isomorphism

$$\varphi : W \otimes V^* \longrightarrow \ell(V, W)$$

defined by

$$\varphi(w \otimes f)(v) = f(v)w, \quad \forall f \in V^*, v \in V, w \in W$$

It suffices to show  $\varphi$  is an  $H$ -module homomorphism.

To this end,

$$\begin{aligned}
\varphi(a \circ (w \otimes f)) &= \varphi(\Delta(a)(w \otimes f))(v) \\
&= \sum_{(a)} \varphi(a_{(1)}w \otimes a_{(2)} \circ f)(v) \\
&= \sum_{(a)} (a_{(2)} \circ f)(v) a_{(1)}w \\
&= \sum_{(a)} a_{(1)}f(S(a_{(2)})v)w \\
&= a_{(1)}\varphi(w \otimes f)(S(a_{(2)})v)
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\Rightarrow \varphi(a \circ (w \otimes f)) &= a_{(1)}\varphi(w \otimes f)S(a_{(2)}) \\
&= a \circ \varphi(w \otimes f), \quad \forall f \in V^*, w \in W, a \in H
\end{aligned}$$

Thus,  $\varphi$  is an  $H$ -module homomorphism.  $\square$

## 2.14 Adjoint action

$H$  itself becomes an  $H$ -module under the adjoint action  $\text{Ad}$  defined by

$$\text{Ad } a \circ b = \sum_{(a)} a_{(1)}bS(a_{(2)}), \quad \forall a, b \in H$$

To see this in fact gives rise to an  $H$ -module structure, we have  $\forall a, b, c \in H$

$$\begin{aligned}
\text{Ad}(ab) \circ c &= \sum_{(ab)} (ab)_{(1)}cS[(ab)_{(2)}] \\
&= \sum_{(a),(b)} a_{(1)}b_{(1)}cS(a_{(2)}b_{(2)}) \\
&= \sum_{(a),(b)} a_{(1)}b_{(1)}cS(b_{(2)})S(a_{(2)}) \\
&= \sum_{(a)} a_{(1)}(\text{Ad } b \circ c)S(a_{(2)}) \\
&= \text{Ad } a \circ (\text{Ad } b \circ c)
\end{aligned}$$

which is the module requirement.

## 2.15 Invariants

Given an  $H$  module  $V$ , we call  $v \in V$  **invariant** if

$$av = \epsilon(a)v, \quad \forall a \in H, v \in V$$

**Lemma 25.1.** *The central elements of  $H$  are precisely the invariants under the adjoint action.*

*Proof.* First by co-associativity, we have

$$\begin{aligned}
(1 \otimes \Delta)\Delta(a) &= \sum_{(a)} a_{(1)} \otimes \Delta(a_{(2)}) \\
&\equiv \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \\
&= (\Delta \otimes I)\Delta(a) \\
&= \sum_{(a)} \Delta(a_{(1)}) \otimes a_{(2)}
\end{aligned}$$

Then, antipode property  $\Rightarrow$

$$\begin{aligned}
\sum_{(a)} a_{(1)} \otimes S(a_{(2)})a_{(3)} &= \sum_{(a)} a_{(1)} \otimes m \circ (S \otimes I)\Delta(a_{(2)}) \\
&= \sum_{(a)} a_{(1)} \otimes \epsilon(a_{(2)}) \cdot 1 \quad (\text{antipode}) \\
&= \sum_{(a)} a_{(1)}\epsilon(a_{(2)}) \otimes 1 \\
&= a \otimes 1 \quad (\text{co-unit}) \tag{2.6}
\end{aligned}$$

Thus, if  $z \in H$  is arbitrary,

$$\begin{aligned}
az &= \sum_{(a)} a_{(1)}zS(a_{(2)})a_{(3)} \\
&= \sum_{(a)} [\text{Ad } a_{(1)} \circ z]a_{(2)}
\end{aligned}$$

Hence if  $z$  is invariant under adjoint action, so

$$\begin{aligned}
\text{Ad } a \circ z &= \epsilon(a)z, \forall a \in H \\
\Rightarrow az &= \sum_{(a)} [\text{Ad } a_{(1)} \circ z]a_{(2)} \\
&= \sum_{(a)} \epsilon(a_{(1)})za_{(2)} \\
&= z \sum_{(a)} \epsilon(a_{(1)})a_{(2)} \\
&= za \quad (\text{co-unit})
\end{aligned}$$

$\Rightarrow z$  is central.

Conversely, if  $z$  is central, it must be an invariant under adjoint action, since  $\forall a \in H$

$$\begin{aligned}\text{Ad } a \circ z &= \sum_{(a)} a_{(1)} z S(a_{(2)}) \\ &= z \sum_{(a)} a_{(1)} S(a_{(2)}) \\ &= \epsilon(a) z \quad (\text{antipode})\end{aligned}$$

Thus  $z \in H$  is invariant  $\iff z \in H$  is central.

□