STAT4404: Advanced Stochastic Processes II, Semester 1, 2013.

Take Home Assignment #5 ,

Last Updated: April 30, 2013.
Due: May 13, 2013, 4:00PM.

Exercise 1

(i) Assume, $X_1, X_2, \ldots$ is an i.i.d. sequence with $\mathbb{E}[X_1^4] < \infty$. During class you saw a proof for the (SLLN):

$$\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{a.s.} 0,$$

in the case where $\mathbb{E}[X_1] = 0$. Generalize the statement and the proof to the case, $\mathbb{E}[X_i] = \mu$.

(ii) Illustrate, by means of simulation, the behavior of the stochastic process,

$$\frac{\sum_{i=1}^{n} X_i}{n},$$

where $\mathbb{E}[X_i]$ is not defined (e.g. for Cauchy random variables).

Exercise 2

This exercise deals with the "standard" Brownian Bridge,

$$B^0(t) = B(t) - tB(1),$$

where $\{B(t)\}$ is a standard Brownian motion process.

(i) Prove in detail that $\{B^0(t),\ t \in [0,1]\} =^d \{B(t),\ t \in [0,1] \mid B(1) = 0\}$, by showing that finite dimensional distributions are the same.

(ii) Prove:

$$\mathbb{P}\Big(\max_{t\in[0,1]} B^0(t) > x\Big) = e^{-2x^2}.$$

To do so you may want to look up a reference for the Brownian Bridge.

(iii) Verify using Monte-Carlo simulation the following,

$$\mathbb{P}\Big(\max_{t \in [0,1]} |B^0(t)| > x \Big) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2 x^2},$$

and illustrate this "nicely".

Exercise 3

This exercise deals with the Kolmogorov-Smirnov Statistic.

(i) Read Section 2.2 in Whitt's book. Expand on the description under equation 2.7, showing equality in distribution between the two random vectors (i.e. justify this further, based on properties of the Poisson process). You may also illustrate it (somehow) using Monte-Carlo simulation, say for $n = 2$ or $n = 3$.

(ii) Assume you are doing a "goodness of fit" test based on the K-S statistic, for an exponentially distributed random sample with mean 2. Find (using extensive Monte Carlo simulations) the distributions of the test statistic (2.5), under the H_0 assumption, for $n = 5, 10, 30, 100$. Compare your curves to the asymptotic value investigated in Exercise 2(iii) above.

Exercise 4

This exercise requires you to complete some convergence proofs not covered during class.

(i) Prove that if $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{p} X$.

(ii) Prove that for independent, $X_1, X_2, \ldots$, if $X_n \xrightarrow{a.s.} c$ (where c is constant) then $X_n \xrightarrow{c.c.} c$.

(iii) Prove that if $X_n \xrightarrow{d} c$ (where c is constant) then $X_n \xrightarrow{p} c$.