STAT4404: Advanced Stochastic Processes II, Semester 1, 2013. Quiz 4 (**with solution**)

Problem 1:

Consider a "telephone booth". Assume people arrive to the booth according to a Poisson process at rate 1 (person/minute) and that phone call durations are exponentially distributed with mean 1. Assume that upon arrival, if the person finds an occupied (in use) booth, she leaves and never returns.

Assume that at time $t = 0$ the phone booth is empty. Let $D(t)$ denote the number of people that have left the booth by time t, after completing their phone calls.

(i) Write an expression for $P\big(D(t) \le n\big)$ for some natural number n.

Solution:

It is clear that $D(\cdot)$ is a renewal process, with inter-renewal time distributed as a sum of two independent, mean 1, exponentials. Such a random variable is a Gamma(2,1), random variable (also known as Erlang(2,1)).

Now, S_n (the sum of the first n renewal times) is distributed as Gamma(2n,1).

Thus,

$$P\big(D(t) \le n\big) = P\big(S_n \ge t\big) = \int_t^\infty \frac{1}{(2n-1)!} x^{2n-1} e^{-x} dx.$$

(ii) Find,

$$\lim_{t \to \infty} P\big(D(t) \le \frac{t}{2}\big).$$

Justify your answer.

Hints: Construct $\{D(t),\ t \ge 0\}$ as a renewal process. To write the solution to (ii) you do not necessarily need to rely on (i).

Solution:

$D(\cdot)$ is a renewal process with inter-renewal mean = 2 and inter-renewal variance = 2 (you get this from summing the mean and variance of two exp(1) random variables – for these the mean is 1 and variance is 1).

Hence from the CLT for renewal processes,

$$\frac{D(t) - 2^{-1}t}{\sqrt{\frac{2}{2^3}t}} \to^d N(0,1).$$

So,

$$\lim_{t \to \infty} P\Big(\frac{D(t) - 2^{-1}t}{\sqrt{\frac{2}{2^3}t}} \le 0\Big) = \frac{1}{2}.$$

but,

$$P\Big(\frac{D(t)-2^{-1}t}{\sqrt{\frac{2}{2^3}t}}\le 0\Big)=P\Big(D(t)\le\frac{t}{2}\Big),$$

hence the desired result is $\frac{1}{2}$.

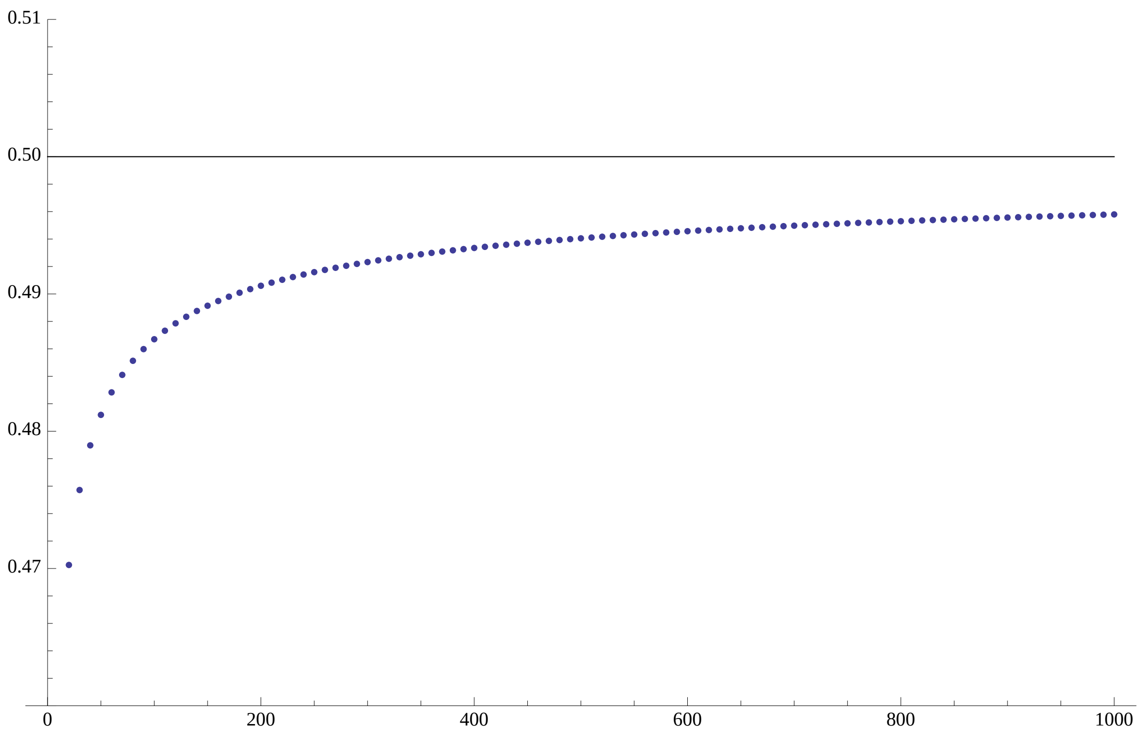
This also agrees, with the answer to (i). E.g. try in Mathematica:

$$p[\text{t\_}]:=\text{NIntegrate}\left[\frac{1}{(t-1)!}x^{t-1}E^{-x},\{x,t,\infty\}\right]$$

The Mathematica function p[] then computes the probability $P(D(t)\le\frac{t}{2})$ for integer t. You can plot it as follows:

ListPlot[
Table[$\{t, p[t]\}, \{t, 10, 1000, 10\}$], PlotRange $\to \{0.46, 0.51\}$, Epilog $\to$ {Line[{{0, 0.5}, {1000, 0.5}}]}]

The obtained plot is:



Problem 2:

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. non-negative random variables. Denote the geometric mean,

$$G_n := \Big(\prod_{i=1}^{n} X_i\Big)^{\frac{1}{n}}.$$

(i) Find sequences a_n, b_n, $n = 1, 2, \ldots$, perhaps depending on the distribution of X_1, such that,

$$\big(a_n G_n\big)^{b_n} \to^d e^Z,$$

where Z is a standard normal random variable. Justify the convergence in distribution. You may assume all moments of X_1 and related random variables are finite.

Solution:
Let's look at $\log \big(a_n G_n\big)^{b_n}$:

$$\log \big(a_n G_n\big)^{b_n} = b_n(\frac{1}{n}\sum_{i=1}^{n} \log X_i + \log a_n).$$

Now since $\log e^Z = Z$ we would like to find the "proper" a_n, b_n such that $\log G_n$ converges to Z based on the CLT. Hence it must be that,

$$b_n = \frac{1}{\sqrt{Var(\log X_1)/n}}, \qquad a_n = e^{-E[\log X_1]}.$$

With such a_n, b_n, by the CLT, $\log \big(a_n G_n\big)^{b_n}$ converges in distribution to Z and then by the continuous mapping theorem, $G_n = e^{\log \big(a_n G_n\big)^{b_n}}$ converges in distribution to e^Z. Note that e^Z is actually called a log-normal random variable (since its log is the normal).

(ii) Consider now for continuous $t \geq 0$

$$G_n(t) := \big(a_{\lfloor nt \rfloor} G_{\lfloor nt \rfloor}\big)^{b_n}, \qquad (*).$$

As $n \to \infty$, does $G_n(t)$ converge in distribution (weakly) to some limiting process, $G(\cdot)$? If so, write that process in terms of a Brownian motion process, $B(\cdot)$.

Hint: Don't be shy to use logarithms.

Solution:
Note: It would have been better to denote, $G_n(t)$ by $\hat{G}_n(t)$, so as not to confuse with the notation of the sub-problem above.
This sub-problem is the "process version" of the above. Look at $\log G_n(t)$:

$$\begin{aligned}\log G_n(t) &= b_n(\log G_{\lfloor nt \rfloor} + \log a_{\lfloor nt \rfloor}) \\ &= \frac{\frac{1}{\lfloor nt \rfloor}\sum_{i=1}^{\lfloor nt \rfloor} \log X_i - E[\log X_1]}{\sqrt{Var(\log X_1)/n}}.\end{aligned}$$

Now by Donsker's theorem, $\log G_n(\cdot)$ converges on (D, J_1) to $B(\cdot)$, a standard brownian motion.

Now applying the mapping $D \to D$, taking $e^{x(t)}$ for every t, of $x(t)$, we have by the continuous mapping theorem that,

$$G_n(\cdot) \to^d e^{B(\cdot)}, \qquad \text{on} \quad (D, J_1).$$