STAT4404: Advanced Stochastic Processes II, Semester 1, 2013. Quiz 3 (**with solution**)

Let $\{B_1(t)\}$ and $\{B_2(t)\}$ be two independent standard Brownian motion processes defined on the same probability space.

Denote,

$$A(n) = \frac{B_1(n) - B_2(n)}{n^\alpha}, \quad n = 1, 2, \ldots, \quad \alpha \geq 0.$$

Problem 1:

Denote,

$$A_0(n) = A(n) - nA(100), \quad n = 0, 1, 2, \ldots, 100.$$

Find $\mathbb{E}\left[A_0(n)\right]$ and $\text{Var}\big(A_0(n)\big)$.

Problem 2:

Describe the convergence of $\{A(n), n = 0, 1, 2, \ldots\}$, prove your findings. That is find when,

$$A(n) \to Y,$$

where the convergence is in one of several ways (almost sure, probability, r'th mean, or distribution). Your answer may depend on α. The element Y may either a constant, a non-degenerate random variable or infinity.

You can assume any theorem stated or proved in class as known.

Solution (for both problems):
We have,

$$\begin{aligned} A(n) &= n^{-\alpha}\big(B_1(n) - B_2(n)\big) \\ &= n^{-\alpha}\Big(\sum_{i=1}^{n}\big(B_1(i) - B_1(i-1)\big) - \sum_{i=1}^{n}\big(B_2(i) - B_2(i-1)\big)\Big) \\ &= n^{-\alpha}\sum_{i=1}^{n}\Delta_1(i) - \Delta_2(i) \\ &= n^{-\alpha}\sum_{i=1}^{n}\tilde{\Delta}(i), \end{aligned}$$

where for $k = 1, 2$ and $i = 1, \ldots, n$,

$$\Delta_k(i) := B_k(i) - B_k(i-1) \qquad \sim N(0,1) \quad \text{i.i.d.},$$

and thus,

$$\tilde{\Delta}(i) := \Delta_1(i) - \Delta_2(i) \qquad \sim N(0,2) \quad \text{i.i.d.}.$$

So we have,

$$A(n) \sim N(0,\ (n^{-\alpha})^2 n2) = N(0, 2n^{1-2\alpha}).$$

Thus observe that for $\alpha < \frac{1}{2}$, $Var(A(n))$ is increasing while for $\alpha > \frac{1}{2}$ it is decreasing. For $\alpha = \frac{1}{2}$, $Var(A(n)) = 2$ for all n.

Problem 1:
First, Showing $E[A_0(n)] = 0$ is trivial, even though, $A(n)$ and $A(100)$ are not independent, we just have,

$$E[A_0(n)] = E[A(n)] - nE[A(100)] = 0 - n0 = 0.$$

To compute the variance, break up to independent increments (in the standard way):

$$\begin{aligned} A_0(n) &= A(n) - nA(100) \\ &= n^{-\alpha}\sum_{i=1}^{n}\tilde{\Delta}(i) - \frac{n}{100^\alpha}\sum_{i=1}^{n}\tilde{\Delta}(i) \\ &= n^{-\alpha}\sum_{i=1}^{n}\tilde{\Delta}(i) - \frac{n}{100^\alpha}\Big(\sum_{i=1}^{n}\tilde{\Delta}(i) + \sum_{i=n+1}^{100}\tilde{\Delta}(i)\Big) \\ &= \big(n^{-\alpha} - \frac{n}{100^\alpha}\big)A(n) - \frac{n}{100^\alpha}\big(A(100) - A(n)\big). \end{aligned}$$

Now $A(n)$ and the increment $A(100) - A(n)$ are independent, so take variance and rearrange to obtain:

$$Var(A_0(n)) = 2n^{1-2\alpha} - \frac{4}{100^\alpha}n^{2-\alpha} + \frac{2}{100^{2\alpha-1}}n^2.$$

Problem 2:

It was only required to give an answer of the form below (or more detailed). A complete characterization of when and how $A(n)$ converges (based on α), is possible, yet was not required in the given time frame.

- First observe that for $\alpha < \frac{1}{2}$, $Var(A(n))$ is increasing and thus it does not converge in any of the modes to any RV.
- Now for $\alpha = 1/2$ the distribution of $A(n)$ is fixed as $N(0, 2)$ thus it converges in distribution to $N(0, 2)$.
- For $\alpha > 1/2$ we have by Chebyshev's inequality that $A(n)$ converges in probability to 0. Similarly we have convergence in the 2'nd mean to 0.
- For $\alpha = 1$ we have by the SLLN that $A(n)$ converges to 0.