STAT4404: Advanced Stochastic Processes II, Semester 1, 2013. Quiz 1

Exercise 1:

Let $Y_0, Y_1, \ldots$ be a sequence of independent, identically distributed random variables such that $E(Y_n) = 0$ and $E(Y_n^2) = \sigma^2 > 0$ for all n. Show that the pair $(Z, \mathcal{F})$ is a martingale with

$$Z_n = \Big(\sum_{j=1}^{n} Y_j\Big)^2 - \sigma^2 n$$

and $\mathcal{F}_n = \sigma(Y_0, \ldots, Y_n)$. Justify every step.

Exercise 2:

Let $Y_0, Y_1, \ldots$ be independent, identically distributed random variables whose moment generating function $\phi(\theta) = E(e^{\theta Y_i})$ is finite for some value $\theta \neq 0$. Show that $(Z, \mathcal{F})$ is a martingale

$$Z_n = Z_n(\theta) = \prod_{j=1}^{n} \frac{e^{\theta Y_j}}{\phi(\theta)} = \frac{e^{\theta S_n}}{\phi(\theta)^n}.$$

Justify every step.

Exercise 3:

Consider the matching problem. For example, suppose N people, each wearing a hat, have gathered in a party and at the end of the party, the N hats are returned to them at random. Those that get their own hats back then leave the room. The remaining hats are distributed among the remaining guests at random, and so on. The process continues until all the hats have been given away. Let X_n denote the number of guests still present after the n^{th} round of this hat returning process. At each round, we expect one person to get his own hat back and leave the room. In other words, $E(X_n - X_{n+1}) = 1$. Find a martingale which is a function of X_n.

Solutions

In the following exercises, we are asked to show that a given pair $(Z, \mathcal{F})$ is a martingale and thus the following conditions must be verified.

1. $E|Z_n| < \infty$
2. $E(Z_{n+1}|\mathcal{F}_n) = Y_n$.

Exercise 1:

Noticing the following

$$(\sum_{i=1}^{n+1} Y_j)^2 = (\sum_{i=1}^{n} Y_j)^2 + Y_{n+1}^2 + 2\sum_{j=1}^{n} Y_j Y_{n+1} \tag{1}$$

we verify the martingale assumptions.

1. $$\begin{aligned} E|Z_n| < E\Big(|(\sum_{j=1}^{n} Y_j)^2 - \sigma^2 n)|\Big) &\leq E\big(|(\sum_{j=1}^{n} Y_j)^2|\big) + \sigma^2 n \\ &\leq nVar(Y_j) + \sigma^2 n \\ &\leq 2n\sigma^2 < \infty \end{aligned}$$

   since $Y_1, Y_2, \ldots$ are i.i.d and $E(Y_i) = 0$ for all $i = 1, \ldots, n$.

2. It follows by (1), the independency of the random variables in the sequence and the moments properties.

$$\begin{aligned} E(Z_{n+1}|Y_0, \ldots, Y_n) &= E\big((\sum_{j=1}^{n+1} Y_j)^2 - \sigma^2(n+1)|Y_0, \ldots, Y_n\big) \\ &= E\big((\sum_{i=1}^{n} Y_j)^2 + Y_{n+1}^2 + 2\sum_{j=1}^{n} Y_j Y_{n+1} - \sigma^2(n+1)|Y_0, \ldots, Y_n\big) \\ &= E\big((\sum_{i=1}^{n} Y_j)^2|Y_0, \ldots, Y_n) + E(Y_{n+1}^2) + 2E(\sum_{j=1}^{n} Y_j Y_{n+1}|Y_0, \ldots, Y_n)\big) - (n+1)\sigma^2 \\ &= (\sum_{i=1}^{n} Y_j)^2 - \sigma^2 n = Z_n. \end{aligned}$$

Exercise 2:

1. Using the Taylor series expansion of $e^{\theta S_n}$ and the equality $E(e^{\theta S_n}) = \prod_{i=1}^{n} E(e^{e^{\theta X_i}})$, we have

$$
\begin{aligned}
E|Z_n| &= E|\frac{e^{\theta S_n}}{\phi(\theta)^n}| \le \frac{1}{|\phi(\theta)^n|}E|e^{\theta S_n}| \\
&\le \frac{1}{|\phi(\theta)^n|}E(\sum_{i=1}^{n}|\theta X_i|^n/n!) \\
&= \frac{1}{|\phi(\theta)^n|}\big(E(\sum_{n=0}^{\infty}(\theta X_i)^{2n}/(2n)!) + E(\sum_{n=0}^{\infty}(|\theta X_i|)^{2n+1)}/(2n+1)!\big) \\
&\le \frac{1}{|\phi(\theta)^n|}E(\sum_{n=0}^{\infty}(\theta X_i)^n/n!) \le 1.
\end{aligned}
$$

2. Considering the properties of the sequence $X = \{X_1, \ldots, X_n\}$, the independence between the random variables and the fact that S_n is measurable in the σ-algebra generated by X, the second assumption holds.

$$
\begin{aligned}
E(Z_{n+1}|X_1,\ldots,X_n) &= E\big(\frac{e^{\theta S_n+1}}{\phi(\theta)^{n+1}}|X_1,\ldots,X_n\big) \\
&= E\big(\frac{e^{\theta S_n+\theta X_{n+1}}}{\phi(\theta)^{n+1}}|X_1,\ldots,X_n\big) \\
&= \frac{e^{\theta S_n}}{\phi(\theta)^{n+1}}E\big(e^{\theta X_{n+1}}|X_1,\ldots,X_n\big) \\
&= \frac{e^{\theta S_n}}{\phi(\theta)^{n+1}}E\big(e^{\theta X_{n+1}}\big) \\
&= \frac{e^{\theta S_n}}{\phi(\theta)^{n+1}}\phi(\theta) \\
&= Z_n.
\end{aligned}
$$

Exercise 3:

We consider the σ-algebra $\mathcal{F}$ generated by $\{X_1, \ldots, X_n\}$ and we search for the martingale by first considering the pair $(X, \mathcal{F})$.

$$
\begin{aligned}
E(X_{n+1}|X_1,\ldots,X_n) &= E(X_{n+1} - X_n + X_n|X_1,\ldots,X_n) \\
&= E(X_{n+1} - X_n|X_1,\ldots,X_n) + X_n \\
&= X_n - 1.
\end{aligned}
$$

Then the sequence $\{X_n + n : n \ge 0\}$ is a martingale:

1.

$$
\begin{aligned}
E(X_{n+1} + (n+1)|X_1,\ldots,X_n) &= X_n + (n+1) - 1 \\
&= X_n + n.
\end{aligned}
$$

2. The number of hats after the n^{th} round, X_n, is smaller or equal than the number of hats at the beginning of matching game.

$$
E|X_n + n| < N + n < \infty.
$$