STAT4404: Advanced Stochastic Processes II, Semester 1, 2013.

Take Home Assignment #7 ,

Last Updated: May 27, 2013.
Due: June 20, 2013.

Exercise 1

Consider a sequence of i.i.d. random variables, $X_0, X_1, X_2, \ldots$. Let $W_0 = 0$ and,

$$W_{n+1} = (W_n + X_n)^+, \quad n = 0, 1, 2, \ldots.$$

Further set, $S_0 = 0$ and, $S_n = \sum_{i=0}^{n-1} X_i$.

(i) Simulate W_n for some sequence $\{X_n\}$ with $\mathbb{E}[X_1] < 0$. Plot you trajectorie(s).

(ii) Prove that,

$$W_n = \max(S_n, S_n - S_1, \ldots, S_n - S_{n-1}, 0) = S_n + \min_{1 \le k \le n} S_k.$$

(iii) Explain in detail, how this stochastic recursive sequence is related to the GI/G/1 queue.

Exercise 2

An approximation for GI/G/1 systems with $\lambda \approx \mu$, yet $\lambda < \mu$, is that the steady state workload, is distributed as an exponential random variable with parameters depending on $\left(\lambda^{-1} - \mu^{-1}\right)$ as well as on the variances of the inter-arrival and service times.

(i) Write out the parameters of this exponential distribution in detail. Loosely justify where it comes from.

(ii) Use 1.(i) above to simulate for some example cases with $\lambda = \mu - \epsilon$ for some small ϵ. Verify the parameters that your wrote in 2.(i) above, e.g. based on the EDF or the sample mean and sample variance. Note that *long* simulation runs are required to obtain good estimates.

(iii) Explain (completely heuristically) why long simulation runs are needed.

Exercise 3

Let $D(t)$ be the departure process of a GI/G/1 queue with finite moments, arrival rate λ and service rate μ. Denote by $B(\cdot)$ a standard Brownian Motion.

(i) Prove that if $\lambda < \mu$ then,

$$\frac{D(n\cdot) - \lambda n\cdot}{\sqrt{n}} \to^d c_1 B(\cdot), \quad \text{on } (D, J_1),$$

for some $c_1 > 0$. Find c_1.

(i) Prove that if $\lambda > \mu$ then,

$$\frac{D(n\cdot) - \lambda n\cdot}{\sqrt{n}} \to^d c_2 B(\cdot), \quad \text{on } (D, J_1),$$

for some $c_2 > 0$. Find c_2.