STAT4404: Advanced Stochastic Processes II, Semester 1, 2013.

Take Home Assignment #6 ,

Last Updated: May 14, 2013.
Due: May 27, 2013, 4:00PM.

Exercise 1

Let Z_n, $n = 1, 2, 3, ...$ be a sequence of i.i.d. random variables with mean μ and finite variance σ^2.

(i) State Donsker's theorem describing the fluctuations of $S_n := \sum_{i=1}^{n} Z_i$.

(ii) Try to illustrate the theorem graphically. Creating a simple animation (overlay of several trajectories) is also an option.

(iii) Consider the "projection map" from $\mathcal{D}$ (the space of RCLL functions) to $\mathbb{R}$, defined as:

$$h_{t_0}\Big(x(\cdot)\Big) = x(t_0).$$

Argue why it is a continuous map. Use the projection map at $t_0 = 1$ to show that the "standard CLT" is a special case of Donsker's theorem.

(iv) The full proof of Donsker's theorem requires more probalisitic details than we have covered. Nevertheless, showing that finite dimensional distributions of,

$$\hat{S}_n(t) := \frac{S_{\lfloor nt \rfloor} - \mu n t}{\sqrt{n}},$$

converge to the finite dimensional distributions of Brownian motion, is not too complicated using the "standard CLT for multi-dimensional vectors". Carry this out.

Exercise 2

This exercise deals (once more) with the Kolmogorov-Smirnov Statistic. Read Section 2.2 in Whitt's book again. Explain in detail how Donsker's theorem and one of the versions of the continuous mapping theorem, yield the asymptotic distribution of the K-S statistic.

Exercise 3

You have seen the FCLT for renewal processes. The more elementary version is the "standard CLT" for a renewal process, $N(t)$. The statement is that the random variables,

$$\frac{N(t) - \lambda t}{\sqrt{\lambda c^2 t}},$$

converge in distribution to a standard normal random variable as $t \to \infty$. Here λ^{-1} is the mean of the inter-renewal time, and c^2 is its squared coefficient of variation (assumed to be finite). Prove this results using the "standard CLT" and elementary considerations (including the fact that $N(t)/t$ converges a.s. to λ).

Exercise 4

Look at pages 244-247 in the paper, "The asymptotic variance of departures in critically loaded queues", Adv. Appl. Prob.43. pp.243-263, 2011.

(i) Explain how a weak convergence result of $\hat{D}_n(\cdot)$ is used to obtain the asymptotic variance rate of the original process. You do not need to explain the weak convergence result of $\hat{D}_n(\cdot)$ in itself, only describe how it is used.

(ii) Follow and understand the details of Theorem 2.1. Assume now $D_L(t)$ is a delayed departure process, constructed as follows: As an item leaves the queue, it is delayed for a random duration with finite mean and variance, generated independently of all other durations. Then $D_L(t)$ counts how many items have left the queue after being delayed. Show that the same asymptotic variance results that hold for $D(t)$, also hold for $D_L(t)$.