STAT4404: Advanced Stochastic Processes II, Semester 1, 2013. Take Home Assignment 2, Due: March 25, 2013, 4:00 P.M.

Exercise 1:

If $(Y, \mathcal{F})$ is a martingale such that $E(Y_n^2) < \infty$ for all n. Show that for $i \leq j \leq k$

- $E\big((Y_k - Y_j)Y_i\big) = 0$ surely.
- $E\big((Y_k - Y_j)^2|\mathcal{F}_i\big) = E(Y_k^2|\mathcal{F}_i) - E(Y_j^2|\mathcal{F}_i)$ almost surely.

Exercise 2:

Let Y be a martingale and let $g : \Re \to \Re$ be a convex function. Show the following:

- $\{g(Y_n) : n \geq 0\}$ is a submartingale given that $E(g(Y_n)^+) < \infty$ for all n.
- $|Y_n|$, Y_n^2 and Y_n^+ are submartingales whenever the appropriate moment conditions are satisfied and give the moment conditions.

Exercise 3:

Consider a random walk with steps $X_1, X_2, \ldots$ where X_i take values 1 or -1 with probability 1/2. If $S_n = \sum_{i=1}^n X_i$ represents the fortune at time n with $S_0 = k$, consider the random sequence $\{S_n : n \geq 0\}$ and assume that there are two absorbing barriers at 0 and N.

- Show that $S_n^2 - n$ is a martingale.
- Using the above martingale and making assumptions similar to those of De Moivre's, find the probability of ruin p_k where $p_k = P(\textit{ being absorbed at } 0|S_0 = k)$ and the expected duration of the game for the gambler's ruin problem. State your assumptions clearly.

Exercise 4:

A TV program organizes a game where 4 people get into a jewelery store and take as many items as they can fit into a backpack of volume v. However only one person wins the game and that is the one who's bag worth the most. Note that the volume and value of the items are independent.

The total number of available items is n, each of the items have volume V_i and it is worth W_i. The random variables V_i and W_i are independent, non negative with finite means and such that $W_i \leq M$ for all i and some fixed $M \in \Re$.

The goal is to maximize the total worth of the the items inside the backpack. In other words, you want to find out variables $z_1, \ldots, z_n$ each of them with values 0 or 1 such that $\sum_{i=1}^{n} z_i V_i \leq v$ and which maximizes $\sum_{i=1}^{n} z_i W_i$. Let Z be the maximal possible worth of the backpack contentsw and show that

$$P(|Z - E(Z)| \geq x) \leq 2\ exp\{-x^2/(2nM^2)\}$$

for $x \geq 0$.

Exercise 5 (Optional):

Consider a graph with n vertices $v_1, \ldots, v_n$ for each $1 \leq i < j \leq n$. Edges between v_i and v_j were placed with probability p and different pairs were joined independently of each other. If two vertices v_i and v_j are connected we say they are *neighbours*. Consider X to be the minimal number of different labels required so that each vertex maybe be labelled differently from each of its *neighbours*. Show that

$$P(|X - E(X)| \geq x) \leq 2\ exp\{-\frac{x^2}{2n}\}$$

for $x \geq 0$.