STAT4404: Advanced Stochastic Processes II, Semester 1, 2013.

Take Home Assignment #1 (updated March 4),

Last Updated: March 4, 2013.

Due: March 19, 2013, 9:00AM.

This assignment deals with single server queueing systems generated by random variables that we refer to as *mixed exponential random variables* (MEXP). This distribution is parameterized by three parameters, $\gamma > 0$, $a \geq 0$, $p \in [0,1]$. If X is a MEXP random variable then with probability p, X follows an $exp(\gamma)$ distribution and with probability $(1-p)$, $X = a$ (a fixed value). I.e. these random variables are mixtures of an exponential and a deterministic value.

In short we shall refer to such random variables as $MEXP(\gamma,\ a,\ p)$.

Exercise 0

- Illustrate the CDF of the MEXP distribution (draw it by hand or with software).
- Calculate the mean of the MEXP distribution.
- Calculate the SCV (squared coefficient of variation = variance / mean^2) of the MEXP distribution.
- Use software to generate $100,000$ random values of $MEXP(2, 1, 3/4)$. Estimate the mean and the SCV from the random variables and compare to your results above. Also, plot the empirical CDF from your large random sample, compare it to your illustration.

In the remainder of the exercises, we deal with

$$MEXP(\gamma_1, a_1, p_1)\ /\ MEXP(\gamma_2, a_2, p_2)\ /\ 1,$$

queueing systems. These are single server queueing system with first come first serve service, in which the inter-arrival times are i.i.d. $MEXP(\gamma_1, a_1, p_1)$ and the service times are i.i.d. $MEXP(\gamma_2, a_2, p_2)$. These include the D/D/1, M/D/1, D/M/1 and M/M/1 systems. They can be viewed as special cases of M/G/1, GI/M/1 and GI/G/1.

Exercise 1

In this exercise we consider D/D/1 queueing systems. I.e. $p_1, p_2 = 0$.

- State the condition for stability of the queueing system in terms of a_1 and a_2.
- If the system is stable, what is the long term fraction of time during which the server is busy.
- Illustrate a trajectory of the queue length and work load processes for $t \in [0, 10]$ choosing some **stable** values of your choice of a_1, a_2 not bigger than 2, and assuming the system is empty at time 0 at which point the first arrival occurs.
- Repeat the previous item, now with some **unstable** parameters (again, not bigger than 2).

Exercise 2

In this exercise we consider M/M/1 queueing systems. I.e., $p_1, p_2 = 1$.

- Write the condition for stability in terms of γ_1 and γ_2. In this case, what is the steady-state distribution of Q?
- The sojourn time distribution, is the distribution of the duration of time it takes a customer arriving in steady state to the system, to leave the system (this can include waiting if she arrives to a non-empty system and otherwise is simply the service time). Prove that the distribution of the sojourn time is, $exp(\gamma_2 - \gamma_1)$.
- A service provider is giving service to a stream of customers arriving according to a Poisson process at a rate of 1 customer per minute on average. The service durations follow an exponential distribution with a mean of $1/\gamma_2$. Having a fast server (γ_2 large) incurs a cost to the service provider. The costs per minute are $\$5\gamma_2$. Further, the service provider has insurance costs for customers that are in the system for a duration of more than 3 minutes. Each such customer costs the server \$7! I.e. periodically, the insurance company gets a listing of the number of customers that have stayed in the system for more than 3 minutes, and charges the service provider \$7 for each such customer.

  Can you formulate an optimization problem that the service provider is facing, in terms of γ_2? What is the solution? I.e. what is the choice of γ_2 that will minimize the service provider's cost?

Exercise 3

In this exercise we consider a form of M/G/1 queueing systems by setting $p_1 = 1,\ p_2 < 1,\ a_2 > 0$.

- State the condition for stability of the queueing system.
- Use K-P formula to write down the PGF of the number of customers in the system in steady state.
- What is the mean number of customers in the system in steady state?
- Try to write an expression for the probability that the number of customers in the system in steady state exceeds K. This may be challenging.

Exercise 4

In this exercise we consider a modification of M/G/1 queueing systems: Assume that when the server begins service after being idle (after a period in which there were no customers in the system), the duration of the service of the first customer to be served follows a different distribution than that of the other customers to be served during the busy period, say $MEXP(\gamma_3, a_3, p_3)$.

- What is the stability condition for this system? Do you think it depends on γ_3, a_3, p_3? Explain your (intuitive) reasoning.
- THIS ITEM IS CANCELED: Follow the development of the K-P formula based on the embedded Markov chain at departure times. Try to derive a similar formula for the PGF of the number of customers in the system in steady state (this time for this modified M/G/1 system where the first service is different).
- Assume $p_1, p_2, p_3 = 1$ (i.e. the system becomes a modification of the M/M/1 queue). In this case, write the balance equations and try to solve them to get the distribution of the number of customers in the system in steady state.

Exercise 5

This exercise is optional (bonus), yet we recommend that you do it. Consider GI/G/1 systems following MEXP distributions for the inter-arrival and service. Write a discrete event simulation program designed to estimate mean queue sizes. Your program can assume that the queues are stable (what is the condition for this?). Your program should take as parameters, $p_i, a_i, \gamma_i,\ i = 1, 2$ and a time horizon T. It should then advance time from $t = 0$ to $t = T$ in discrete jumps based on the events of the queueing system (inter-arrival and/or service completion). As this occurs, the program should evaluate,

$$\bar{Q}(t) := \frac{\int_0^t Q(s)ds}{t},$$

to get an estimate of the steady state queue length mean. The resulting $\bar{Q}(T)$ should yield an estimate of mean queue lengths. You can assume that $p_i > 0$ to avoid D/D/1 type cases.

- Use your program to verify the M/M/1 and M/G/1 results that you obtained above, for a few example values.

- In general, there is NO exact solution for the steady state mean queue lengths of GI/G/1 queues! Thus your program is of real value. Use it to estimate the steady state queue mean lengths of,

  $$MEXP(1, 1, 3/4)\ /\ MEXP(2, 2, 3/4)\ /\ 1,$$

  queues. Run the simulation 100 times, each time for $T = 100,000$ to obtain 90% confidence intervals for your estimate of the mean queue size.