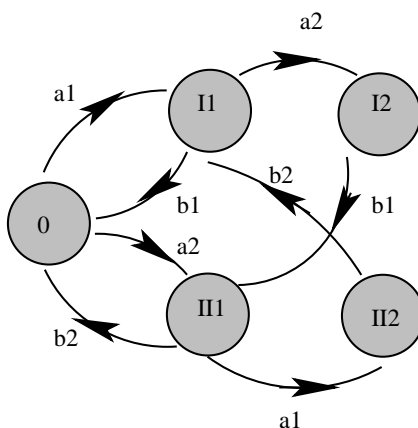


Probability Models and Stochastic Processes

Solutions to Assignment 9

- Suppose in the Repairman Problem (Example 10.3 of the notes) there is one repairman but there are two different machines. Both have exponential life and repair times, but the rates are different, say a_1, a_2 for the lifetimes, and b_1, b_2 for the repair times, respectively. Describe the failure-repair process by means of a suitable Markov process, and give its transition rate graph.

Solution: For this problem, we need to keep track of which machine failed, and can no longer consider just the number of failed machines. Let us label our machines I and II. Let states I_1 and I_2 represent 1 and 2 failed machines where machine I failed first, and similarly for II_1 and II_2 . Let $(X_t, t \geq 0)$ be a process with state space $E = \{0, I_1, I_2, II_1, II_2\}$, describing the number *and* order of failed machines. Since all the life and repair times are exponential, (X_t) is a Markov process with transition rate graph given below.



[4]

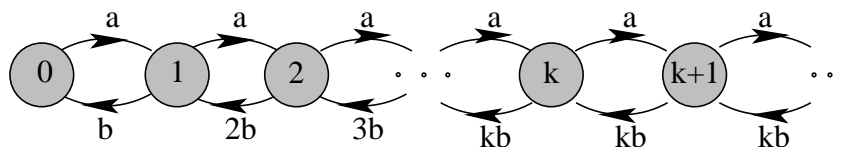
- In an M/M/k-queueing system customers arrive at a single queue according to a Poisson process with rate a , say. There are k servers who serve these customers on a first-come-first-serve basis. Customers who cannot be served right away have to wait in a single queue. The service times have an exponential distribution with mean $1/b$. All service and interarrival times are independent. Let X_t be the number of customers in the whole system at time t .

- Argue why $(X_t, t \geq 0)$ is a Markov process. Give its transition rate graph.
- Let Y_t be the number of customers at time t waiting in the queue to be served. Argue whether or not $(Y_t, t \geq 0)$ is also a Markov process. If it is, give its transition rate graph.

Solution:

- Suppose there are i customers in the system at time t . If $i \leq k$, all are being served with iid exponential service times. If $i > k$, only k are being served. Due to the memoryless property of the exponential distribution, each of the $m = \min(i, k)$ service times at time t is again iid $\text{Exp}(b)$. The time a customer will leave the system is the minimum of m service times, which is $\text{Exp}(mb)$

distributed. Since arrivals occur according to a Poisson process with rate a , interarrival times are $\text{Exp}(a)$ distributed. Thus the number of people in the system at time t , $(X_t, t \geq 0)$, is a Markov process, with transition rate graph given below.



[4]

- (b) We argue not as follows. Suppose $X_0 = 0$. The expected sojourn time until the *first* transition from state 0 to 1 (corresponding to X_t travelling from 0 to $k+1$) will differ from *subsequent* such transitions (corresponding to X_t travelling from k to $k+1$). In particular this means that sojourn times in state 0 need not be exponential (since if (X_t) jumps more than once in $\{0, 1, \dots, k\}$ the waiting time will be Gamma distributed). Thus (Y_t) is not a Markov process. [3]
3. A mouse is let loose in the maze of Figure 1, and moves from each compartment to an adjacent one with equal probability, independent of the past. The mouse spends an exponential time in each compartment: for each of 1, 3, 4 the mean time is one second, and for 2, 5, 6 it is two seconds. Let $(X_t, t \geq 0)$ be the Markov process that describes the position of the mouse for times $t \geq 0$. Assume the mouse starts in compartment 1 at time $t = 0$.

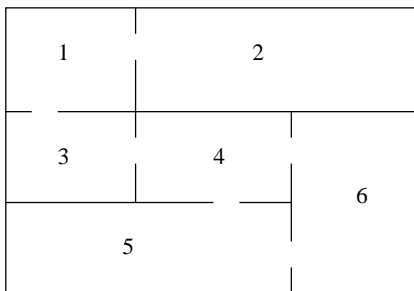


Figure 1: A maze

- (a) Write down the transition rate matrix Q for the process (X_t) .
- (b) Give the transition matrix K of the corresponding jump chain $\{Y_n\}$.
- (c) Find the long-term mean time spent between moves.

Solution:

- (a) In this case, the transition rate matrix Q is:

$$Q = \begin{pmatrix} -1 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & -1 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/4 & -1/2 & 1/4 \\ 0 & 0 & 0 & 1/4 & 1/4 & -1/2 \end{pmatrix}.$$

[4]

- (b) From Q , or directly from the question, the transition matrix K for the jump chain is:

$$K = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

[2]

- (c) The jump chain is finite, irreducible (as all compartments are accessible from all others), and aperiodic (since, for example, we can move from compartment 4 to 4 in two or three steps). Hence the limiting distribution $\boldsymbol{\nu}$ is unique and stationary, and can be found by solving $\boldsymbol{\nu}K = \boldsymbol{\nu}$. This gives solution $\boldsymbol{\nu} = (2, 1, 2, 3, 2, 2)/12$. The expected time in seconds spent in the compartments are $\mathbf{s} = (1, 2, 1, 1, 2, 2)$. Thus the long-term average time spent between moves is just $\boldsymbol{\nu}\mathbf{s}^T = (2 + 2 + 2 + 3 + 4 + 4)/12 = 17/12$ seconds. [3]

Total [20]