

# Probability Models and Stochastic Processes

## Solutions to Assignment 8

1. Let  $(N_t)$  be a Poisson process with rate  $\lambda$  and  $N_0 = 0$ . Show that:

(a) for  $0 \leq u \leq t$  and  $j = 0, 1, 2, \dots, n$ ,  $\mathbb{P}(N_u = j | N_t = n) = \binom{n}{j} \left(\frac{u}{t}\right)^j \left(1 - \frac{u}{t}\right)^{n-j}$ ,  
i.e. the conditional distribution of  $N_u$  given  $N_t = n$  is  $\text{Bin}(n, u/t)$ .

(b) for  $s, t \geq 0$ ,  $\mathbb{E}[N_t N_{t+s}] = \lambda t (1 + \lambda(t + s))$ .

*Solution:*

(a) We have  $\mathbb{P}(N_u = j | N_t = n) = \mathbb{P}(N_u = j, N_t = n) / \mathbb{P}(N_t = n)$ . For  $0 \leq u \leq t$ ,  $\mathbb{P}(N_u = j, N_t = n) = \mathbb{P}(N_u = j) \mathbb{P}(N_{t-u} = n - j)$ . Thus

$$\mathbb{P}(N_u = j | N_t = n) = \frac{(\lambda u)^j e^{-\lambda u} (\lambda(t-u))^{n-j} e^{-\lambda(t-u)} n!}{j!(n-j)! (\lambda t)^n e^{-\lambda t}},$$

which simplifies directly to  $\binom{n}{j} \left(\frac{u}{t}\right)^j \left(1 - \frac{u}{t}\right)^{n-j}$ . [3]

(b) We have

$$\begin{aligned} \mathbb{E}[N_t N_{t+s}] &= \mathbb{E}[\mathbb{E}[N_t N_{t+s} | N_t]] = \mathbb{E}[N_t \mathbb{E}[(N_t + N(t, t+s)) | N_t]] \\ &= \mathbb{E}[N_t(N_t + N_s)] = \mathbb{E}[N_t^2] + \mathbb{E}[N_t] \mathbb{E}[N_s] \\ &= \text{Var}(N_t) + \mathbb{E}[N_t]^2 + \lambda t \lambda s = \lambda t + \lambda^2 t^2 + \lambda t \lambda s \\ &= \lambda t (1 + \lambda(t + s)). \end{aligned}$$

[3]

2. Imagine that you are a worker bee whose task is to collect pollen. In an exceedingly unrealistic model, the bee modeling department posits that on a particular day in Spring, the positions of flowering spotted gum trees and flowering iron bark trees can be described by a pair of spatial Poisson processes with rates

$$\lambda(x, y) = \frac{1 - [\cos(x^2 + y^2)]^2}{1 + \sqrt{x^2 + y^2}} \quad \text{and} \quad \nu(x, y) = \frac{1 - [\sin(x^2 + y^2)]^2}{1 + \sqrt{x^2 + y^2}}$$

respectively, where  $x$  and  $y$  are measured in kilometers from your beehive. According to this model, what is:

- (a) the probability you have to fly more than 1 km to reach a flowering tree?
- (b) the probability that there are exactly 5 flowering trees within 1 km?
- (c) the expected number of flowering trees within 9 km?

*Solution:* Let  $(N_{x,y})$  and  $(M_{x,y})$  be our two spatial Poisson processes. Firstly, convert to polar coordinates, so that  $r = \sqrt{x^2 + y^2}$  and  $\theta = \arctan(y/x)$ , giving  $\lambda(r, \theta) = \frac{1 - [\cos(r^2)]^2}{1 + r}$  and  $\nu(r, \theta) = \frac{1 - [\sin(r^2)]^2}{1 + r}$ , neither of which depend in  $\theta$ . Secondly, notice that the number of flowering trees of any kind in a region  $A \subseteq \mathbb{R}^2$  is Poisson with parameter  $\iint_A r \lambda(r, \theta) dr d\theta + \iint_A r \nu(r, \theta) dr d\theta = \iint_A r \mu(r, \theta) dr d\theta$ , where  $\mu(r, \theta) = (1 + r)^{-1}$ . (The factor  $r$  comes from the polar transformation, giving  $dx dy \rightarrow r dr d\theta$ .) [2]

- (a) Let  $D = \{(r, \theta) : r \leq 1\}$ , so this probability is  $\mathbb{P}(T(D) = 0) = \exp(-\xi)$ , where  $\xi = \int_0^{2\pi} \int_0^1 r \mu(r, \theta) dr d\theta = 2\pi(1 - \log(2))$ . Hence  $\mathbb{P}(T(D) = 0) = \exp(-2\pi(1 - \log(2))) \approx 0.1454$ . [2]
- (b) In the same vein,  $\mathbb{P}(T(D) = 5) = \xi^5 \exp(-\xi)/5! \approx 0.03229$ . [2]
- (c) The number of points within a nine kilometer radius has a Poisson distribution with rate  $\zeta = \int_0^{2\pi} \int_0^9 r \mu(r, \theta) dr d\theta = 2\pi(9 - \log(10)) \approx 42.0811$ . The expected number is then just  $\zeta$ . [2]

3. Imagine that today is the start of a new financial crisis, and as a consequence people are keen to withdraw cash from their local branch. Suppose that customers arrive at a particular bank at the times of a Poisson process  $(N_t)$  with time-dependent rate function

$$\lambda(t) = \exp(t), \quad 0 \leq t \leq 8,$$

where  $t$  is measured in hours after the opening time of 8:30 am. The bank closes its doors to customers at 4:30 pm. Let  $X_i$  be the amount that the  $i$ th customer withdraws, and assume that  $X_1, X_2, \dots$  is a sequence of i.i.d.r.v.'s independent of  $(N_t)$ . The amount of money in the bank at time  $t$  is denoted by  $Y_t$  (this is a *compound Poisson process*). We have  $Y_t = Y_0 - X_1 - X_2 - \dots - X_{N_t}$ .

- (a) What is the probability that a total of 3000 customers arrive during business hours, with exactly 30 arriving before noon?
- (b) If the bank initially has three million dollars in cash, calculate the expected amount of cash the bank has after today's bank run if  $\mathbb{P}(X_1 = \$1000 \times k) = \exp(-1)/k!$  for  $k = 0, 1, \dots$

*Solution:*

- (a) The day can be broken into two time-intervals, namely 8:30 am till noon, and noon till 4:30 pm. The number of customers in the first time-interval has a Poisson distribution with rate  $\alpha = \int_0^{3.5} \lambda(t) dt = \exp(3.5) - 1 \approx 32.1155$ . The number in the second interval is Poisson with rate  $\beta = \int_{3.5}^8 \lambda(t) dt = \exp(8) - \exp(3.5) \approx 2947.8425$ , independently of the first. The probability in question is thus

$$\frac{\alpha^{30} \exp(-\alpha)}{30!} \times \frac{\beta^{2970} \exp(-\beta)}{2970!} \approx 0.0004557.$$

[3]

- (b) Now  $\mathbb{E}[Y_8] = \mathbb{E}[Y_0 - \sum_{k=1}^{N_8} X_k] = Y_0 - \mathbb{E}[\mathbb{E}[\sum_{k=1}^{N_8} X_k | N_8]] = Y_0 - \mathbb{E}[N_8] \mathbb{E}[X_1]$ . Further,  $X_1$  has the same distribution as  $\$1000 \times Z$ , where  $Z \sim \text{Poi}(1)$ , thus  $\mathbb{E}[X_1] = \$1000 \mathbb{E}[Z] = \$1000$ . Also,  $N_8$  has a Poisson distribution with parameter  $\int_0^8 \lambda(t) dt = \exp(8) - 1 \approx 2979.9580$ , so  $\mathbb{E}[Y_8] = \$3000000 - \$1000 \times (\exp(8) - 1) \approx \$20042$ . [3]

Total [20]