Assignment Number 3

Problem 1 Let (X, d) be a complete metric space, and let $T : X \to X$ be a *contraction mapping*, i.e., there exists $\theta \in [0, 1)$ with

$$d(T(x), T(y)) \leq \theta d(x, y) \qquad \text{for all } (x, y \in X).$$

(Note: T is not necessarily linear.) Show that there exists precisely one $x \in X$ with $T(x) = x$. Does the statement reamain true if the completeness requirement is omitted?

Problem 2 a) For $p = 1, \frac{2}{3}, 2, 3, 10, \infty$, sketch the "unit p-ball" w.r.t. $\| \cdot \|_p$ in $\mathbb{R}^2$.
b) Let $x \mapsto |x|$ und $x \mapsto |x|'$ be two norms on a $\mathbb{K}$-vectorspace V, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Show the equivalence of the following statements:
(i) There exist c and C with $0 < c < C < \infty$, such that: $c|x| \leq |x|' \leq C|x|$.
(ii) In every open $| \cdot |$-ball there is a $| \cdot |'$-ball, and vice versa.
(iii) The topologies induced by $| \cdot |$ and $| \cdot |'$ coincide.
(iv) A sequence $\{x_i\}$ converges to a point x w.r.t. $| \cdot |$ iff $\{x_i\}$ converges to x w.r.t. $| \cdot |'$.

Problem 3 Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded (and nonempty), with smooth boundary, and let $\alpha \in (0, 1]$ be fixed. A function $f : \Omega \to \mathbb{R}$ is called *uniformly Hölder continuous* on Ω (with exponent α), if there exists $K < \infty$ with

$$\sup_{x,y \in \Omega,\ x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq K \qquad (*).$$

The space of all such functions is denoted by $C^\alpha(\overline{\Omega})$. If$(*)$ only holds on every compact subset $\Omega' \subseteq \Omega$, then f is called *locally Hölder continuous* on Ω (with exponent α), denoted $f \in C^\alpha(\Omega)$; note that here, K may depend on Ω'.
a) Show: $C^\alpha(\overline{\Omega}) \subset C^0(\overline{\Omega})$. Show by virtue of an example that the inclusion is strict.
b) Prove: $C^\alpha(\overline{\Omega}) \subset C^\beta(\overline{\Omega})$, but $C^\alpha(\overline{\Omega}) \neq C^\beta(\overline{\Omega})$ for $0 < \beta < \alpha \leq 1$. (The analogous statements hold for $C^\alpha(\Omega)$ and $C^\beta(\Omega)$).
c) We define:

$$[f]_{\Omega,\alpha} := \sup_{x,y \in \Omega,\ x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Is $[\,\cdot\,]_{\Omega,\alpha}$ a norm on $C^\alpha(\overline{\Omega})$? A seminorm?

Due: Thursday, 21/4/2005 before the tutorial

Current assignments will be available at
`http://www.maths.uq.edu.au/courses/MATH4401/Tutorials.html`