MATH4401: Advanced Analysis

First Semester 2003
17/3/2004

Assignment Number 2

Problem 1 Let X be a metric space. Show that there holds:
a) $|d(x,z) - d(y,z)| \le d(x,y)$ (the *reverse triangle inequality*).
b) $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$ defines a metric on X which is equivalent to d (i.e., $\tilde{d}$ is a metric on X, and $x_i \to x$ with respect to d is equivalent to $x_i \to x$ with respect to $\tilde{d}$). What happens if d is an extended metric?

Problem 2 Let (X, τ) be a compact topological space. Show:
a) If Y is Hausdorff and $f : X \to Y$ is continuous, then $f(X)$ is a compact subset of Y.
b) If $f : X \to \mathbb{R}$ is continuous, then there exists $x_0 \in X$ with $f(x) \le f(x_0)$ for every $x \in X$ (i.e., f attains its maximum).
c) If Y is Hausdorff and $f : X \to Y$ is continuous and bijective, then f^{-1} is continuous (and hence f is a homeomorphism).

Problem 3 Let X be the set of all functions $f : [0,1] \to \mathbb{R}^n$ which satisfy $f(0) = 0$ and $|f(x) - f(y)| \le |x - y|$. Given $\varphi \in C(\mathbb{R}^n, \mathbb{R})$, define $\Phi : X \to \mathbb{R}$ via

$$\Phi(f) := \int_0^1 \varphi(f(t))\, dt.$$

Show:
a) X is a compact subset of $C([0,1], \mathbb{R}^n)$ (Hint: Arzela-Ascoli).
b) $\Phi : X \to \mathbb{R}$ is continuous.
c) Use Problem 2b) to show that there exists at least one function $f_0 \in X$ with $\Phi(f) \ge \Phi(f_0)$ for every $f \in X$.

Due: Thursday, 7/4/2004 before the tutorial

Current assignments will be available at
`http://www.maths.uq.edu.au/courses/MATH4401/Tutorials.html`