Q1. (a) Derive Hamilton's equations of motion from Lagrange's equations of motion.

(3 marks)

Lagrange's equations of motion are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{where } L(q_i, \dot{q}_i, t).$$

Now define the conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{and these equations become}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \quad \text{(1)}$$

If

$$H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, q_i, p_i, t),$$

then

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial \dot{q}_i} + p_j \frac{\partial q_j}{\partial q_i} - \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial q_i} = -\frac{\partial L}{\partial q_i} \quad \text{(2)}$$

Also

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + p_j \frac{\partial q_j}{\partial p_i} - \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial p_i} \quad \text{(3)}$$

$$\Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{Hamilton's eqns of motion.}$$

$$\Rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{(1) + (2)}$$
Q1. (b) Show that the following Hamiltonians are integrable and give their integrals of motion.

(i) \[ H(q, p, t) = \frac{p^2}{2} + \cos(q) + e^{-t} \]

(ii) \[ H(\theta_1, \theta_2, I_1, I_2) = I_1^2 + 2I_1I_2 + I_2^2 + I_1I_2 \cos(\theta_1 - 3\theta_2) \]

(7 marks)

i) Since this is a 4 dof system integrability is ensured if there is one constant of the motion. Here that is \( H = \frac{p^2}{2} + \cos(q) \) which is an equally good Hamiltonian for the system since \( e^{-t} \) is a complete differential.

Alternatively use a generating function \( F = qP + e^t \) to transform to "new" coordinates \( P = p, Q = q \) with new Hamiltonian \( H = H + \frac{3F}{3t} \).

ii) Transform to new conjugate variables with \( \Phi_1 = \theta_1 - 3\theta_2 \) and \( \Phi_2 = \theta_2 \) via:

\[ F_2(\theta_1, J_1) = J_1(\theta_1 - 3\theta_2) + J_2 \theta_2 \Rightarrow I_1 = J_1, I_2 = -3J_1 + J_2 \]

The Hamiltonian

\[ H(\Phi_1, J_1) = H_0(\Phi_1) + J_1(3J_1 - 3J_2) \cos \Phi_1 \]

is now independent of \( \Phi_2 \) \( \Rightarrow \frac{\partial H}{\partial \Phi_2} = 0 \Rightarrow J_2 \) constant of motion.

So, for this 2 dof system, there are 2 constants of the motion: \( H \) and \( J_2 = I_2 + 3I_1 \)

\( \Rightarrow \) integrable.

(Actually \( \{J_2, H\} = 0 \) \( \Rightarrow \) completely integrable.)
Q2. Give the primary resonance conditions for the following Hamiltonian.

\[ H(\theta_1, \theta_2, I_1, I_2) = 2I_1^2 + I_2^2 + \epsilon(I_1^2 \sin(\theta_1)^2 \cos(\theta_2) + I_2^2) \]

If \( H \) is fixed as 1, which tori of the unperturbed system, that is the system with \( \epsilon = 0 \), will be resonant?

Assuming you are not close to resonance give the near identity generating function that removes all the oscillating terms to order \( \epsilon \). Hence give the approximate integrals of motion in the original variables.

(10 marks)

\[
\sin \alpha \sin^2 \theta_1 \cos \theta_2 = \frac{\cos \theta_2}{2} - \frac{(\cos(\theta_1 + \theta_2) + \cos(2\theta_1 - \theta_2))}{4} 
\]

There are three resonance conditions (primary)

\[
\frac{\partial H_0}{\partial I_2} = 0, \quad 2\frac{\partial H_0}{\partial I_1} + \frac{\partial H_0}{\partial I_2} = 0 \quad \text{and} \quad 2\frac{\partial H_0}{\partial I_1} - \frac{\partial H_0}{\partial I_2} = 0
\]

\[\Rightarrow \quad I_2 = 0, \quad 4I_1 + 2I_2 = 0 \quad \text{and} \quad 4I_1 - 2I_2 = 0.\]

If \( H_0 = 2I_1^2 + I_2^2 = 1 \) then these imply that

\[
\left\{ I_1 = \pm \frac{1}{\sqrt{2}}, \, I_2 = 0 \right\}, \quad \left\{ I_1 = \pm \frac{1}{\sqrt{6}}, \, I_2 = \pm \frac{2}{\sqrt{6}} \right\}, \quad \left\{ I_1 = \frac{1}{\sqrt{6}}, \, I_2 = \frac{2}{\sqrt{6}} \right\}
\]

Usually the action is taken as \( > 0 \)

\[\Rightarrow \left\{ I_1 = \frac{1}{\sqrt{2}}, \, I_2 = 0 \right\}, \quad \text{No soln} \quad \left\{ I_1 = \frac{1}{\sqrt{6}}, \, I_2 = \frac{2}{\sqrt{6}} \right\} \]
\[ H = 2I_1^2 + I_2^2 + \varepsilon (I_2^2 + \frac{I_1^2}{4}(2\cos \theta_2 - 2\cos(\theta_1 + \theta_2))) \]

Use a near identity generating function

\[ F_2 (\theta_1, \theta_2) = \theta_1 J_1 + \theta_2 J_2 + \varepsilon (g_1 \sin \theta_2 + g_2 \sin (2\theta_1 - \theta_2) + g_3 \sin (2\theta_1 + \theta_2)) \]

to remove all the oscillating terms to order \( \varepsilon \),

where \( g_i (J_i) \)

\[ I_1 = J_1 + \varepsilon (g_2 \cos (2\theta_1 - \theta_2) - g_3 \cos (2\theta_1 + \theta_2)) \]
\[ I_2 = J_2 + \varepsilon (g_1 \cos \theta_2 + 2g_2 \cos (\theta_1 - \theta_2) + 2g_3 \cos (\theta_1 + \theta_2)) \]

To order \( \varepsilon \) the Hamiltonian becomes

\[ H = 2I_1^2 + I_2^2 - \varepsilon 4J_1 (g_2 \cos (\theta_1) + g_3 \cos (\theta_1)) \]
\[ + \varepsilon 2J_2 (g_1 \cos \theta_2 + 2g_2 \cos (\theta_1 - \theta_2) + 2g_3 \cos (\theta_1 + \theta_2)) \]
\[ + \varepsilon \left( J_2^2 + \frac{J_1^2}{4} (2\cos \theta_2 - \cos (\theta_1) - \cos (\theta_2)) \right) \]

So choose \( 2g_1 J_2 = -\frac{J_2^2}{4} \)

\[ 2g_1 J_2 = -\frac{J_2^2}{4} \implies g_1 = \frac{J_2^2}{16J_2} \]

\[ 4(J_1 + 4J_2)g_2 = \frac{J_2^2}{4} \implies g_2 = \frac{J_2^2}{16J_2} \]

\[ 4(J_1 + J_2)g_3 = \frac{J_1^2}{4} \implies g_3 = \frac{J_1^2}{16(J_1 + J_2)} \]

The approximate integrals of motion are \( J_1 \) and \( J_2 \).

\[ J_1 = I_1 + \varepsilon \left( \frac{I_1^2}{16(I_2 - I_1)} \cos (2\phi_1 - \phi_2) - \frac{I_1^2}{16(I_2 + I_1)} \cos (2\phi_1 + \phi_2) \right) \]

Question 3 see next page.
Q3. (a) Describe what happens as you perturb the following twist map

\[ I_{n+1} = I_n \]

\[ \theta_{n+1} = \theta_n + 2\pi \omega(I_n), \quad \text{mod}(1) \]

if \( \omega = \frac{N}{M} \), given that \( \omega(I_n) \) is continuous function and the map remains area preserving. 

(5 marks)

If the unperturbed map is

\[
\begin{pmatrix}
\theta_{n+1} \\
I_{n+1}
\end{pmatrix} = T
\begin{pmatrix}
\theta_n \\
I_n
\end{pmatrix} =
\begin{pmatrix}
\theta_n + 2\pi \omega(I_n) \\
I_n
\end{pmatrix}
\]

Then per \( \omega = \frac{N}{M} \) \( T^M \) is the identity.

If \( \omega > \frac{N}{M} \) then \( T^M \) twists anticlockwise

But for \( \omega < \frac{N}{M} \) \( T^M \) twists clockwise

Now Perturb this to \( T_\epsilon \). The relative twists will be preserved, i.e. further out the twist is anticlockwise but further in """" clockwise.

So on each radial line there is a point which is not twisted!

Let the curve of these points be \( R_\epsilon \)

Consider \( T^M_\epsilon R_\epsilon \), each point moves radially.

But area is preserved so

\[ T^M_\epsilon R_\epsilon \cap R_\epsilon \neq \emptyset \]


unfact they must intersect in an even no of points.

Further if \( x \) is an intersection point then so is \( T_\epsilon x \Rightarrow \exists \) at least \( 2kM \) fixed points of \( T^M_\epsilon \) or period-M points of \( T_\epsilon \).

Question 3 continued on next page.
Q3. (b) Show that the following map

\[ u_{n+1} = u_n - A \sin(2\pi \phi_n) \]

\[ \phi_{n+1} = \phi_n + (u_{n+1})^2 \mod(1) \]

is a product of two involutions and find any lines or curves which are fixed under the involutions of the map. Using this or otherwise investigate the possibility of symmetric period-2 points where one iterate lies on the line \( \phi_n = 0 \).

(5 marks)

\[ \text{Find the involutions:} \]

\[ I_1 (\phi, u) = (-\phi, \frac{-u}{u - A \sin 2\pi \phi}) \]

\[ I_1^2 (\phi) = (-(-\phi), \frac{-u}{u - A \sin 2\pi (-\phi)}) = (\phi, u) \]

\[ I_2 (\phi, u) = (-\phi + u^2, u) \]

\[ I_2^2 (\phi) = (-(-\phi + u^2) + u^2, u) = (\phi, u) \]

such that \( I_2 I_1 \) in the map:

\[ I_2 I_1 (\phi, u) = \left( (-\phi + (u - A \sin 2\pi \phi)^2), \frac{-u}{u - A \sin 2\pi \phi} \right) \]

Fixed points of \( I_1 \)

\( \phi = -\phi \) and \( u = u - A \sin 2\pi \phi \Rightarrow \phi = 0 \) or \( \frac{1}{2} \)

are fixed lines

Fixed points of \( I_2 \)

\( \phi = -\phi + u^2 \) \( \Rightarrow u^2 = 2\phi \mod 1 \) \( \Rightarrow \phi = \frac{u^2}{2} \)

\( \phi = \frac{1}{2} + \frac{u^2}{2} \) \( \) \( \) \( \) \( \) \( \) \( \)

There is a symmetric period-2 point at

\( u_0 = \frac{1}{\sqrt{2}} \), \( \phi_0 = 0 \) \( \Rightarrow u_{n+1} = \frac{1}{\sqrt{2}} \), \( \phi_{n+1} = \frac{1}{2} \)

\( u_{n+2} = \frac{1}{\sqrt{2}} \), \( \phi_{n+2} = 1 / \mod(1) = 0 \).

There are various ways to get this, directly by considering period-2 pts with \( \phi_n = 0 \),

or by using properties of involutions.

Question 4 see next page.