1 For each of the mechanical systems described below, give the number of degrees of freedom of the configuration space.

a) Three juggling pins.
b) A spherical pendulum, consisting of one point mass hanging from a rigid massless rod attached to a fixed support. The pendulum bob may move in any direction subject to the constant imposed by the rigid rod. The point mass is subject to the uniform force of gravity.
c) A point mass sliding without friction on a rigid curved wire.

2 Derive Lagrange’s equations for the following systems.

a) A particle of mass, \( m \), moves in a two-dimensional potential \( V(x, y) = (x^2 + y^2)/2 + x^2y + y^3/3 \), where \( x \) and \( y \) are rectangular coordinates of the particle.
b) A free particle constrained to move on a sphere of radius \( R \), where the generalised coordinates are the angles \( \theta \); the colatitude of the particle and \( \phi \) the longitude.

3 Show that the action of a free particle moving in a constant-velocity straight line path is
\[
\frac{m(x_b - x_a)^2}{2(t_b - t_a)}
\]
where \( x_b = x(t_b) \) and \( x_a = x(t_a) \).

4* Formulate a Lagrangian for a spherical pendulum subject to uniform gravity. What symmetries can you find? Find coordinates that express that symmetry and analytic expression(s) for the conserved quantity.

5 For each of the following Lagrangians derive the Hamiltonian and Hamilton’s equations of motion.

a) \( L(t; x, y, p_x, p_y) = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} - V(x, y) \)
b) \( L(t; \theta, \phi, \alpha, \beta) = \frac{mR^2(\alpha^2 + \beta^2 \sin^2(\theta))}{2} \), where \( \alpha = \dot{\theta} \) and \( \beta = \dot{\phi} \).
c) \( L(x, \dot{x}) = m_0c^2 \left( 1 - \left( 1 - \frac{\dot{x}^2}{c^2} \right)^{\frac{1}{2}} \right) - \frac{1}{2} m_0 \Omega^2 x^2 \) The relativistic Lagrangian for a particle of rest mass \( m_0 \) moving along the \( x \)-axis under the simple harmonic potential field \( V = \frac{1}{2} m_0 \Omega^2 x^2 \).

6* Sketch the curves where the Hamiltonian \( H = c \), a constant for the system with Lagrangian
\[
L(q, \dot{q}) = \frac{\dot{q}^2}{2} - \frac{\alpha^2(1 - q^2)^2}{4}
\]
Indicate whether \( c < 0 \), \( 0 < c < \frac{\alpha^2}{4} \) or \( \frac{\alpha^2}{4} < c \) for the curves that you have drawn.
The Hamiltonian for Kepler’s Central Force Problem is

\[ H(r, \theta, p_r, p_\theta) = \frac{p_r^2}{2\mu} - \frac{k}{r} + \frac{p_\theta^2}{2\mu r^2}, \]

where \( l = r^2 \dot{\theta} \).

a) Write the equations of motion for Kepler’s central force problem.

b) What symmetries can you find?

c) Give analytical expressions for the conserved quantity.

d) Using the equations of motion and the conserved quantity derive a second order equation for \( r(\theta) \) and solve for \( r \). (It may be useful to let \( u = \frac{1}{r} \).)

e) \( \frac{1}{r} = C(1 + e \cos(\theta - \theta_0)) \) describes a hyperbola if \( e > 1 \) and an ellipse if \( 0 < e < 1 \). Use this to determine for what values of \( H r(\theta) \) is a circle, ellipse, parabola or hyperbola.

8* Prove the result that for a Legendre Transformation

\[ \frac{\partial F}{\partial w} + \frac{\partial G}{\partial w} = 0, \]

where \( w \) is a passive variable.

9 For each of the following functions find the function that is related to the given function by the Legendre transform on the indicated active argument. Show that the Legendre transform relations hold for your solution.

a) \( F(x) = ax + bx^2 \)

b) \( F(x, y) = a \sin x \cos y \), with \( x \) active.

c) \( F(x, y, \dot{x}, \dot{y}) = x\dot{x}^2 + 3\dot{x}\dot{y} + y\dot{y}^2 \), with \( \dot{x} \) and \( \dot{y} \) active.

10* Prove that the Poisson Bracket of two constants of the motion is itself a constant of the motion.

11 The motion of the Henon Hiles System is confined to the inside of the region

\[ V(x, y) = x^2 y - \frac{y^3}{3} + \frac{x^2 + y^2}{2} = E. \]

(Here \( \epsilon \) has been taken as zero.) Sketch the contours of \( V(x, y) \). Try \( V = \frac{1}{6} \) and \( V \) small first.

Extra Marks!! can be gained by also solving this problem.

12* Find the function \( z(x) \) that minimises the functional

\[ T(z) = \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1 + z'^2}{z}} \, dx. \]

This is the time taken for a particle moving in the \((x, z)\) plane, starting at \( z = 0 \), where \( z \) is taken as pointing downwards, to fall under gravity along a wire with the path \( z(x) \) from \( x = 0, z = 0 \) to \( x = a, z = b \). So you are working out the shape of wire that gives the shortest time. Note you cannot work out \( z(x) \), the best you can get is a curve written in parametrized form.