1. There are two primary resonances with the following resonance conditions.

The \( \cos 2(\theta_1 - \theta_2) \) gives the resonance condition

\[
\frac{2(\partial H_0)}{\partial I_1} - \frac{2\partial H_0}{\partial I_2} = 0
\]

Here \( \partial H_0 = 1 - 2I_1 - 3I_2 \) and \( \partial H_0 = 1 - 3I_1 + 2I_2 \)

\( \Rightarrow \)

\[
1 - 2I_1 - 3I_2 - 1 + 3I_1 - 2I_2 = 0
\]

\( \Rightarrow \)

\[
I_1 = 5I_2
\]

The \( \cos (3\theta_1 - 2\theta_2) \) term gives the resonance condition

\[
\frac{3\partial H_0}{\partial I_1} - \frac{2\partial H_0}{\partial I_2} = 0
\]

\( \Rightarrow \)

\[
3(1 - 2I_1 - 3I_2) - 2(1 + 3I_1 + 2I_2) = 0
\]

\( \Rightarrow \)

\[
1 - 13I_2 = 0 \quad \text{or} \quad I_2 = \frac{1}{13}
\]

To investigate the resonance at \( I_1 = 5I_2 \)

first remove the other oscillating term, assuming that \( I_2 \) is not close to \( \frac{1}{13} \).

Then use \( F_2 \) generating function to transform to new canonical variables \( (\phi_i, J_i) \rightarrow (\phi_i, J_i) \)

with \( \phi_1 = \theta_1 - \theta_2 \) and \( \phi_2 = \theta_2 \) and \( F(\theta_1, \theta_2, J_1, J_2) \)

Then since \( \phi_1 = \frac{\partial F_2}{\partial J_1} \), \( \phi_2 = \frac{\partial F_2}{\partial J_2} \) \( \Rightarrow \)

\[
F_2 = J_1(\theta_1 - \theta_2) + J_2 \theta_2
\]

So that \( I_1 = \frac{\partial F_2}{\partial \theta_1} = J_1 \) and \( I_2 = \frac{\partial F_2}{\partial \theta_2} = J_2 - J_1 \)

In these new variables the approximate Hamiltonian becomes

\[
H(\phi_i, J_i) = H(\phi_1, \phi_2, J_1, J_2 - J_1)
\]

\[
= J_1 + (J_2 - J_1) - J_1^2 - 3J_1(J_2 - J_1) + (J_2 - J_1)^2
\]

\[+ \epsilon J_1 (J_2 - J_1) \cos 2\phi_1 + O(\epsilon^2)\]
\[ H(\Phi_1, J_1) = J_2 + 3J_1^2 - 5J_2 + J_2^2 + \epsilon J_1(J_2 - J_1) \cos 2\Phi_1 + O(\epsilon^2) \]

In these variables, the resonance condition becomes

\[ \frac{\partial H}{\partial J_1} = 0 \quad \text{i.e.} \quad 6J_1 - 5J_2 = 0 \]

Also to this order \( J_2 \) is a constant of the motion because

\[ J_2 = \frac{\partial H}{\partial \Phi_2} = 0. \]

So near resonance \( J_1 \approx \frac{5}{6}J_2 \) or \( J_{10} = \frac{5}{6}J_2 \)

\[ \Rightarrow \quad \text{let} \quad J_1 = \frac{5}{6}J_2 + \Delta J_1 \quad \text{with} \Delta J_1 \text{ considered small.} \]

Perturbing off this value

\[ H_0(J_1, J_2) = H_0\left(\frac{5}{6}J_2, J_2\right) + \frac{\partial H_0}{\partial J_1}\left(\frac{5}{6}J_2, J_2\right) \Delta J_1 \]

\[ + \frac{1}{2} \Delta J_1^2 \frac{\partial^2 H_0}{\partial J_1^2}\left(\frac{5}{6}J_2, J_2\right) + \ldots \]

\[ = J_2 + \frac{13}{12}J_2^2 + 3\Delta J_1^2 + \ldots \]

The approximate Hamiltonian can be thought of as a function of \((\Phi_1, \Delta J_1)\) as canonical variables.

(The coordinate transform \(J \Rightarrow \text{const} + \Delta J \text{ in canonical.})

\[ H_0(\Phi_1, \Delta J_1) = J_2 + \frac{13}{12}J_2^2 + 3\Delta J_1^2 + \epsilon \frac{5}{36}J_2^2 \cos 2\Phi_1 \]

which is a pendulum with the following phase portrait

\[ \text{The separatrix is given by} \quad \Phi_1 \quad \text{or} \quad \epsilon \frac{5}{36}J_2^2 = 3\Delta J_1^2 + \epsilon \frac{5}{36}J_2^2 \cos 2\Phi_1 \]

So the separatrix width is found by calculating \(|\Delta J_1|\)

at its maximum on the separatrix; say at \(\Phi_1 = \pi/2\)

\[ 3\Delta J_1^2 = 2 \times \frac{5}{36}J_2^2 = \frac{5}{18}J_2^2 \]

The width is twice this

\[ 2 \times \sqrt{\frac{5\epsilon J_2^2}{18 \times 3}} = \frac{J_2}{3} \sqrt{\frac{10\epsilon}{3}} = 2J_{10} \frac{2}{\sqrt{3}} \]
To investigate the resonance at \( I_2 = \frac{1}{13} \) remove the
\[ \varepsilon J_1 I_2 \cos 2(\phi_1 - \phi_2) \] term and let
\[ \phi_1 = \phi_2 + \frac{2 \phi_2}{3} \quad , \quad \phi_2 = \phi_2 \]

\[ \Rightarrow \quad F_2 = J_1 (\phi_1 - \frac{2 \phi_2}{3}) + J_2 \phi_2 \]

and \( I_1 = J_1 \) , \( I_2 = J_2 - \frac{2 \phi_2}{3} J_1 \) and as before
\( J_2 \) is a constant to this order and the new Hamiltonian is
\[ \bar{H}(\phi_1, J_1) = J_1 + J_1^2 \frac{2 \phi_2}{3} - J_2 \frac{2 \phi_2}{3} J_1 + J_2^2 \]
\[ + \varepsilon J_1 (J_2 - \frac{2 \phi_2}{3} J_1)^{\frac{3}{2}} \cos 3 \phi_1 \]

Now the resonance condition is \( \frac{\partial \bar{H}}{\partial J_1} = \frac{1}{3} + 2J_1 \frac{2 \phi_2}{3} - J_2 \frac{2 \phi_2}{3} J_1 = 0 \)

or \( J_{10} = - \frac{3}{26} + \frac{3}{2} J_2 \) \( \left( I_2 = \frac{1}{13} \right) \)

So let \( J = J_{10} + \Delta J \)

\[ \Rightarrow \frac{\partial \bar{H}}{\partial \phi_1} = \text{constant} \quad + \frac{13}{9} \Delta J^2 + \varepsilon \left( -\frac{3}{26} + \frac{3}{2} J_2 \right) \Delta J \frac{9}{2} \cos 3 \phi_1 \]

\[ \Delta J \]

As before separation width
\[ = 2 \times \sqrt{\frac{2 \phi_2}{13} \varepsilon \left( -\frac{3}{26} + \frac{3}{2} J_2 \right) \Delta J} \times 2 \]
\[ = \frac{6}{13} \sqrt{2 \varepsilon \left( -\frac{3}{26} + \frac{3}{2} J_2 \right) \Delta J} \times 13 \]

or
\[ = 2 \times \sqrt{\frac{4 \phi_2}{13} \varepsilon I_{10} \left( \frac{1}{13} \right)^{\frac{3}{2}}} \]
\[ = \frac{12}{13} \sqrt{\frac{\varepsilon I_{10}}{\sqrt{13}}} \]
2. a-e) are two degree of freedom systems where the Hamiltonians are time independent.

A two degree of freedom system is integrable if there are two (linearly independent) integrals, say \( L_1, L_2 \) s.t. \( \{ L_1, L_2 \} = 0 \).

Here since the Hamiltonian is independent of time it is a constant of the motion:
\[
\left( \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} + \{ L_1, H \} = \frac{\partial H}{\partial t} = 0 \right)
\]

So you only need to find one other constant of the motion.

a) Integrable

Let \( \phi_1 = t, \phi_2 = 2t, \phi_3 = 3t \)

Then using an \( F_2 \) generating function
\[ F_2 = J_1 (\phi_1 - \phi_2) + J_2 (2\phi_1 + 3\phi_2) \]
\[ \Rightarrow I_1 = J_1 + 2J_2 \quad \text{and} \quad I_2 = -J_1 + 3J_2 \]

gives a new Hamiltonian
\[
\tilde{H} (\phi_1, J_i) = H (\phi_1 (\phi_i), J_i) \]
\[
= 6J_2 + \cos 2\phi_1 + \cos \phi_2
\]

Either, since \( H \) is not a function of \( J_1 \)
\[ \Rightarrow \frac{\partial H}{\partial J_1} = \phi_1 \Rightarrow \phi_1 \text{ is a constant of the motion.} \]

Also \[ \{ \phi_1, H \} = -\frac{\partial H}{\partial J_1} = 0 \]

OR. \( \tilde{H} \) is separable \( = H_1 + H_2 \)

where \( H_1 = \cos 2\phi_1 \) and \( H_2 = 6J_2 + \cos \phi_2 \)

are individually constants of the motion, because
\[ \{ H_1, H_2 \} = \{ H_1, H_1 \} + \{ H_1, H_2 \} = \{ H_1, H_2 \} \]
\[
= \frac{\partial H_1}{\partial \phi_1} \frac{\partial H_1}{\partial J_1} = 0 \quad \frac{\partial H_1}{\partial \phi_1} \frac{\partial H_1}{\partial J_2} = 0 \quad \frac{\partial H_1}{\partial \phi_1} \frac{\partial H_1}{\partial \phi_2} = 0
\]
\[
= 0
\]
2 b) This Hamiltonian is not integrable.

Although we could make the transformation used in part a) \( \bar{H}(\phi_1, J_1) = 6J_2 + \epsilon (J_1 + 2J_2) (-J_1 + 3J_2)(\cos^2 \phi_1 + \cos \phi_1) \)

It does not separate and is still a function of \( J_1 \).
However, the Hamiltonian for \( H_0 \) is zero so the Hamiltonian does not satisfy the requirements for the KAM theorem. (In fact, the first oscillating term is always resonant and the second never.) So there will be no KAM tori as in the KAM theorem.

c) Once again this Hamiltonian is not integrable.

However, the Hamiltonian for \( H_0 \)

\[
\det \begin{pmatrix}
\frac{\partial^2 H_0}{\partial I_1^2} & \frac{\partial^2 H_0}{\partial I_1 \partial I_2} \\
\frac{\partial^2 H_0}{\partial I_2^2} & \frac{\partial^2 H_0}{\partial I_1 \partial I_2}
\end{pmatrix} = \det \begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix} \neq 0
\]

and \( H_1 \) is a smooth function of the angles.

So the resonances will be isolated and for small enough \( \epsilon \) KAM tori will be present.

d) Integrable because \( \phi_2 \) is not explicitly present in the Hamiltonian and so

\[
\dot{I}_2 = -\frac{\partial H}{\partial \phi_2} = 0 \Rightarrow I_2 \text{ is a second constant of the motion and } \{ I_2, H \} = -\frac{\partial H}{\partial \phi_2} = 0.
\]

e) Integrable because if you change variables to \( \phi_1 = 3\phi_1 - 4\phi_2 \) \( \phi_2 = \phi_2 \)

\[
F_2 = J_1 (3\phi_1 - 4\phi_2) + J_2 \phi_2
\Rightarrow I_1 = 3J_1, \quad I_2 = J_2 - 4J_1
\]

\[
H = 3J_1 (J_2 - 4J_1) + \epsilon 3J_1^2 (\cos \phi_1 + \cos 2\phi_1)
\]

and so \( J_2 \) is a constant of motion as in part d)
2. (f) - (g) are 1½ degree of freedom systems. Since the Hamiltonian is time dependent it is not obviously a constant of the motion. One constant of the motion implies that they are integrable.

(f) Not integrable.
Even after using double angle formulae \( \cos^2(\theta + wt) = \frac{1}{2}(1 + \cos(2\theta + wt)) \) there are two different resonance conditions. However the Hamiltonian \( H_0 = J \) is linear so that the resonances will not be isolated and we cannot expect to find KAM tori for \( \varepsilon \) small.

(g) Transforming to a rotating frame, we can show that this is an integrable system.
Let \( \phi = \theta + wt \) and use a time dependent \( F_2 \) generating function \( F_2 = J(\theta + wt) \)

\[ \Rightarrow I = J \] but the Hamiltonian is changed.
The new Hamiltonian \( K = H + \frac{\partial F_2}{\partial t} \)

\[ K(\phi, J) = J^3 + \varepsilon J (\cos \phi + \sin \phi) + J \omega \]
Since this is time independent \( K \) is a constant of the motion.

(h) Integrable
\[ H(t, \theta, I) = I^2 + \varepsilon \left( \sin(2\theta + 3wt) + \sin wt \right) \]

Now use \( F_2 = J(2\theta + 3wt) + \varepsilon \cos wt \)

\[ \Rightarrow \phi = 2\theta + 3wt, \ I = J \]

To transform to the new Hamiltonian
\[ K = J^2 + \frac{\varepsilon}{2} \sin \phi + \frac{\varepsilon \sin wt}{2} + 3wJ - \frac{\varepsilon \sin wt}{2} \]
which is a constant of the motion.
3. a) \(a_0 = \left[ \frac{21}{13} \right] = 1 \Rightarrow w_1 = \frac{13}{8} \) and \(a_1 = \left[ \frac{13}{8} \right] = 1 \)

\(w_2 = \frac{8}{5} \Rightarrow a_2 = 1, w_3 = \frac{5}{3} \Rightarrow a_3 = 1, \)

\(w_4 = \frac{3}{2} \Rightarrow a_4 = 1, w_5 = 2 \Rightarrow a_5 = 2 \) and \(a_6 = \infty \)

\[12 \left\{ 1, 1, 1, 1, 1, 1, 2, \infty \right\} \]

b) Now \(2 < \sqrt{5} < 3 \)

\(a_0 = 1 \Rightarrow w_1 = \frac{1}{\frac{47-\sqrt{5}}{38}} = \frac{38}{9-\sqrt{5}} = \frac{9+\sqrt{5}}{2} \)

\(a_1 = 5 \Rightarrow w_2 = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2} \)

\(a_2 = 1 \Rightarrow w_3 = \frac{2}{\sqrt{5}-1} \) and so the process will repeat itself ad infinitum

\[\frac{47-\sqrt{5}}{38} \left\{ 1, 1, 1, 1, \ldots \right\} \Rightarrow a_n = 1 \text{ for } n \geq 2. \]

c) \(x_t = 1 \pm \sqrt{2} \)

For \(x_+ = 1+\sqrt{2} \quad a_0 = 2 \quad w_1 = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{1} \)

and so the process will repeat

\(x_+ = 1+\sqrt{2} = \left\{ 2, 2, 2, \ldots \right\} \)

For \(x_- = 1-\sqrt{2} < 0 \):

So consider \(\sqrt{2}-1 \) for which \(a_0 = 0 \)

\(w_1 = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{1} = x_+ \)

\(\Rightarrow x_- = - \left\{ 0, 2, 2, \ldots \right\} \)

d) \(y = \frac{1+\sqrt{5}}{2} \quad a_0 = \left[ \frac{1+\sqrt{5}}{2} \right] = 1 \)

\(w_1 = \frac{1}{\frac{1+\sqrt{5}}{2}-1} = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{4} = \frac{\sqrt{5}+1}{2} = y \)

So \(w_n = \frac{\sqrt{5}+1}{2} \quad \forall \ n \geq 1 \) and \(a_n = 1 \quad \forall \ n \geq 0 \)
4. Since the map is area preserving det \( T = 1 \)
   We are given that \( \text{tr} \, T = \pm 2 \).
   So eigenvalues \( \lambda^2 + \text{tr} \, T \lambda + \det T \)
   given by \( \lambda^2 \pm 2\lambda + 1 = 0 \)
   \( \Rightarrow \lambda = -1 \) twice \( \lambda = 1 \) twice
   So in normal form \( u_{n+1} = \pm u_n, v_{n+1} = \pm v_n \)
   In \( \bigoplus \) case all points are fixed.
   \( \bigotimes \) case " ... " are period-2.
   except the origin which is fixed.

5. See notes for this one.

6. \( \frac{1}{\gamma^2} = \frac{1}{(1 + \nu^5)^2} = \frac{3 - \sqrt{5}}{2} \Rightarrow a_0 = 0 \)

   \( \omega_1 = \frac{2}{3 - \sqrt{5}} = \frac{3 + \sqrt{5}}{2} \Rightarrow a_1 = 2 \)

   \( \omega_2 = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2} \Rightarrow a_2 = 1 \)

   and \( a_n = 1 \) for \( n \geq 2 \)

   \( \frac{1}{\gamma^2} = [0, 2, 1, 1, 1, ... ] \)

   Rational Approximates are \( [0, \infty] = 0 \),
   \( [0, 2, -\infty] = \frac{1}{2 + \frac{1}{\infty}} = \frac{1}{2} \),
   \( [0, 2, 1, \infty] = \frac{1}{2 + \frac{1}{1 + \frac{1}{\infty}}} = \frac{1}{3} \),
   \( [0, 2, 1, 1, \infty] = \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\infty}}}} = \frac{2}{5} \),
   \( [0, 2, 1, 1, 1, \infty] = 3/5 \),

   If \( K < 0 \) then \( I_{n+1} \approx I_n \) and winding number = \( \frac{I_n}{2\pi K} \)

   So \( I_n = 0, \frac{1}{4\pi}, \frac{1}{6\pi}, \frac{2}{10\pi} \) and \( \frac{3}{16\pi} \) when \( K = 0 \).
6. The $Q$ dependence is more complicated however you can show that an elliptic period $-M$ point lies on the line $Q = 0$

![Graph](attachment:image.png)

7. a) Since $I_1 x = x$ and $I_1(I_2 I_1)^n x = (I_2 I_1)^n x$

\[
(I_2 I_1)^{2n} x = (I_2 I_1)^{n-1} I_2 I_1 (I_2 I_1)^n x
= (I_2 I_1)^{n-1} I_2 (I_2 I_1)^n x
= (I_2 I_1)^{n-1} I_1 (I_2 I_1)^n x
= (I_2 I_1)^{n-1} I_2 (I_2 I_1)^n x
= \cdots
= I_2 (I_2 I_1) x
= I_1 x = x
\]

b) let $x_0 = \begin{pmatrix} 0 \\ J \end{pmatrix}$ and $n = 2$ then $(I_2 I_1)^n x_0$ lies on the line $Q = 0$ or $Q = \pi$

But \[
(I_2 I_1)^2 \begin{pmatrix} 0 \\ J \end{pmatrix} = \begin{pmatrix} 2J - k \sin J \\ J \end{pmatrix}
\Rightarrow 2J - k \sin J = 0 \text{ or } \pi
\]

If we set $2J - k \sin J = 0 \Rightarrow J = 0$ which is a fixed point and not a period-4 pt. So set $2J - k \sin J = \pi$ and solve for $J$. (For instance if $k = 0.3$ then $J = 0.2736112 \times 2\pi$.)