Chapter 8

A semimartingale market model

In this chapter we shall apply the theory of stochastic calculus and differential equations to a selection of economic problems.

Semimartingale market models are mentioned in Øksendal (1998), Karatzas & Shreve (1991), Klebaner (1998). These books have a focus on stochastic calculus and touch on finance as an application of the theory. Texts such as Lamberton & Lapeyre (1996) and also Musiela & Rutkowski (1997) have the focus reversed.

Martingales have been a recurring theme throughout this course. As early as Chapter 3 we saw the risk-neutral valuation formula emerge as a convenient way to price contingent claims. We noted then the occurrence of the martingale measure $\mathbb{Q}$ in this formula. And we stated that this $\mathbb{Q}$-dependence wasn't a coincidence - there was something deeper going on. The theory of the intermediate chapters has now put us into a position to justify this statement. Indeed, the two most important results presented in this chapter are Theorem 22, and Theorem 23 on the relationship between $\mathbb{Q}$ and arbitrage-free and complete markets.

We shall restrict attention to market models which are, in fact, Itô processes. We'd like to point out, however, that more general semimartingale market models are currently being researched. General semimartingale market models include those with discontinuities such as the jump diffusion described in Section 7.1.4.

Modify the market model we've used throughout to now include $d \geq 1$ risky assets. Thus our stock market now contains a single riskless asset B_t and d risky

assets $(S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)})$. The bond B_t is assumed to follow the ODE

$$\frac{dB_t}{dt} = r_t B_t, \quad B_0 = 1.$$

In contrast to the discrete time case, where we adjusted our portfolio only at the 'ticks' of the clock, in this model we shall assume that the portfolio is rebalanced continuously. The rebalancing may only take into account the information about the stock price S_t up to the current time t, for this reason we choose to work with an Itô process representation

$$dS_t^{(i)} = \mu_t^{(i)} dt + \sum_{j=1}^{m} \sigma_t^{(ij)} dW_t^{(j)}, \quad S_0^{(i)} = s_i,$$

where μ_t and σ_t satisfy the appropriate conditions on integrands and

$$W = \left(W_t = (W_t^{(1)}, W_t^{(2)}, \ldots, W_t^{(m)})^t; t \geq 0 \right)$$

is an m-dimensional Brownian motion. For example S_t could follow the stochastic differential equation $dS_t = \mu S_t dt + \sigma S_t dW_t$ as suggested by Black & Scholes (1973). We shall frequently refer to the discounted stock price process defined as $\tilde{S} = (\tilde{S}_t = B_t^{-1} S_t; t \geq 0)$.

8.1   Martingales and Arbitrage

Recall that $\mathbb{Q}$ is the probability measure chosen such that the discounted stock price process $\tilde{S}$ is a martingale. Also recall that the risk-neutral valuation formula provides a convenient means with which to price any contingent claim. If C is any claim which follows the risk neutral valuation formula then

$$B_t^{-1} C_t = \mathbb{E}_{\mathbb{Q}} \left(B_T^{-1} C_T | \mathcal{F}_t \right), \quad 0 \leq t \leq T.$$

Actually, by virtue of Exercise 6.3.1, this means that the discounted process $\tilde{C}_t = B_t^{-1} C_t$ is also a $\mathbb{Q}$-martingale. Furthermore, according to the following theorem, the claim C may be replicated by a portfolio containing the underlying stocks and bonds.

Theorem 21 (Martingale representation theorem). *Suppose that M is an $\mathcal{F}_t$-martingale with volatility $\sigma_t > 0$. If N is any other $\mathcal{F}_t$-martingale, then there exists a previsible ϕ such that $\int_0^T \phi_t^2 \sigma_t^2 dt < \infty$ and*

$$N_t = N_0 + \int_0^t \phi_s dM_s.$$

Furthermore ϕ is unique apart from scalar multiples.

Essentially the martingale representation theorem says that if there is a measure $\mathbb{Q}$ and filtration $\mathcal{F}_t$ under which M_t is a martingale, then any other martingale (w.r.t. the same measure and filtration) can be expressed in terms of M_t. It implies that if $\tilde{C}$ and $\tilde{S}$ are both $\mathbb{Q}$-martingales, then there must be a process ϕ such that $d\tilde{C}_t = \phi_t d\tilde{S}_t$ for all t. Thus C is (theoretically[1]) replicated by holding ϕ_t units of stocks and $\psi_t = B_t^{-1}(C_t - \phi_t S_t)$ bonds for all $t \leq T$.

Now suppose the existence of a claim V whose value at time t, doesn't necessarily satisfy the risk-neutral valuation formula. That is to say, there is a time $t < T$ when

$$V_t \neq B_t \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1} V_T \middle| \mathcal{F}_t \right).$$

In other words, we are assuming existence of a commodity, able to be traded within the market, whose discounted process $B_t^{-1} V_t$ is *not* a martingale under the same measure $\mathbb{Q}$ which makes the discounted stock price process $\tilde{S}$ a martingale. We can define a new process U with value following

$$U_t = B_t \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1} V_T \middle| \mathcal{F}_t \right).$$

By the martingale representation theorem, U may be constructed in terms of stocks and bonds, and therefore it must also be a tradable asset. Thus we have two commodities U and V whose values agree at expiry, but at times $t < T$ we possibly have $U_t \neq V_t$. If there is to be no arbitrage in the market, then this is an impossibility. This contradiction gives rise to the following result.

Theorem 22. *If there exists a martingale measure $\mathbb{Q}$ for the discounted stock price process $\tilde{S}$, then the market has no arbitrage.*

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[1]see Chapter 4 for an overview of the issues involved in hedging in continuous time

Perhaps Theorem 22 is at a level of abstraction too far removed from reality. It would be useful to know what it means in terms of the parameters $\mu_t = (\mu_t^{(1)}, \mu_t^{(2)}, \ldots, \mu_t^{(d)})^t$ and $\sigma_t = [\sigma_{ij}; i = 1, \ldots, d; j = 1, \ldots, m]$. According to Girsanov's Theorem (Theorem 15), the m-dimensional process $\tilde{W}_t = \int_0^t \mathbf{u}_s ds + W_t$ will be a Brownian motion under $\mathbb{Q}$ defined implicitly as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \mathbf{u}_s dW_s - \frac{1}{2}\int_0^t \mathbf{u}_s^2 ds\right). \tag{8.1}$$

Thus finding a $\mathbb{Q}$ under which $\tilde{S}$ is a martingale is the same as finding a $\mathbf{u}$ such that for $i = 1, \ldots, d$

$$
\begin{aligned}
d\tilde{S}_t^{(i)} &= d\left(B_t^{-1} S_t^{(i)}\right), \\
&= B_t^{-1}\left(\mu_t^{(i)} - r_t S_t^{(i)} - \sum_{j=1}^m \sigma_t^{(ij)} u_t^{(j)}\right) dt + B_t^{-1} \sum_{j=1}^m \sigma_t^{(ij)} d\tilde{W}_t^{(j)},
\end{aligned}
$$

is driftless. Therefore, to check that the market is arbitrage-free we could check that there exists an m-dimensional $\mathbf{u}_t$ such that

$$\mu_t - r_t S_t = \sigma_t \mathbf{u}_t. \tag{8.2}$$

Example 32. *Consider a market model with $B_t = 1, \forall t$, and which contains two risky assets driven by 2-dimensional Brownian motion $W_t = \left(W_t^{(1)}, W_t^{(2)}\right)$*

$$
\begin{aligned}
dS_t^{(1)} &= \mu dt + \sigma dW_t^{(1)}, \\
dS_t^{(2)} &= \nu dt + \rho dW_t^{(2)}.
\end{aligned}
$$

This model is arbitrage-free if there is a solution to

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \rho \end{pmatrix}\begin{pmatrix} u_t^{(1)} \\ u_t^{(2)} \end{pmatrix},$$

which there obviously is.

Example 33. *Consider a market model with $dB_t = r\,B_t dt$, $r > 0$, and three risky assets driven by a 2-dimensional Brownian motion*

$$
\begin{aligned}
dS_t^{(1)} &= 2dt + dW_t^{(1)}, \\
dS_t^{(2)} &= dt + 2dW_t^{(2)}, \\
dS_t^{(3)} &= dt + dW_t^{(1)} - 2dW_t^{(2)},
\end{aligned}
$$

This system is arbitrage-free if there is a solution to

$$
\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - r \begin{pmatrix} S_t^{(1)} \\ S_t^{(2)} \\ S_t^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_t^{(1)} \\ u_t^{(2)} \end{pmatrix},
$$

which there is since $S_t^{(3)} = S_t^{(1)} - S_t^{(2)}$.

Example 34. *Consider a market model with $B_t = 1, \forall t$, and two risky assets driven again by 2-dimensional Brownian motion*

$$
\begin{pmatrix} dS_t^{(1)} \\ dS_t^{(2)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} dW_t.
$$

The system

$$
\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u_t^{(1)} \\ u_t^{(2)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
$$

has no solutions. So the market may have an arbitrage opportunity. Indeed the value process of the portfolio $\phi = \left(\psi, \phi_t^{(1)}, \phi_t^{(2)} \right)$ is given by

$$
\begin{aligned}
V_t &= V_0 + \int_0^t \psi dB_s + \int_0^t \phi_s^{(1)} dS_s^{(1)} + \int_0^t \phi_s^{(2)} dS_s^{(2)}, \\
&= V_0 + \int_0^t 2\phi_s^{(1)} + \phi_s^{(2)} ds + \int_0^t \phi_s^{(1)} - \phi_s^{(2)} dW_s^{(1)} + \int_0^t 2\phi_s^{(2)} - 2\phi_s^{(1)} dW_s^{(2)}.
\end{aligned}
$$

If we choose $\phi_t^{(1)} = \phi_t^{(2)} = k$, then

$$
V_t = V_0 + 3kt.
$$

Thus, choosing ψ such that $V_0 = 0$, the portfolio ϕ is would make wealth out of nothing.

8.2   Martingales and complete markets

Not only are martingales and instances of arbitrage very much connected, but we can also decide if a given market is complete by examining the martingale measure $\mathbb{Q}$. A market is *complete* if every contingent claim is attainable in the sense that it can be replicated by a self-financing trading strategy consisting of the underlying securities. Complete markets were briefly mentioned in Section 2.1.

Mathematically we can describe a complete market as follows. Let

$$\phi_t = \left(\psi_t, \phi_t^{(1)}, \phi_t^{(2)}, \ldots, \phi_t^{(d)} \right)$$

represent a portfolio. That is to say $\phi = (\phi_t; t \geq 0)$ is a trading strategy which, at time t, involves holding ψ_t bonds and $\phi_t^{(i)}$ units of stock i $(i = 1, \ldots, d)$. At this time the value of the portfolio is simply

$$V_t = \psi_t B_t + \sum_{i=1}^{d} \phi_t^{(i)} S_t^{(i)}. \tag{8.3}$$

A strategy is *self-financing* if no money is brought in or taken away from the portfolio after it has been set up. In other words, the gains and losses the portfolio makes are solely due to gains and losses on the underlying investments $\left(B_t, S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)} \right)$. If the portfolio is only rebalanced at discrete time points t and $t+h$, and there is no infusion or withdrawal of funds, then

$$V_{t+h} - V_t = \psi_t \left(B_{t+h} - B_t \right) + \sum_{i=1}^{d} \phi_t^{(i)} \left(S_{t+h}^{(i)} - S_t^{(i)} \right).$$

In continuous-time the analogous condition is

$$dV_t = \psi_t dB_t + \sum_{i=1}^{d} \phi_t^{(i)} dS_t. \tag{8.4}$$

Note that this condition is different from the expression which would be obtained for dV_t if we applied Itô's formula to (8.3).

A market is complete if we can find a replicating portfolio ϕ, satisfying (8.4), for every (European) contingent claim. If C_T is the payoff function, then we require

$$
\begin{aligned}
C_T &= V_0 + \int_0^T dV_t, \\
&= V_0 + \int_0^T \psi_t dB_t + \sum_{i=1}^d \int_0^T \phi_t^{(i)} dS_t^{(i)}.
\end{aligned}
\tag{8.5}
$$

Completeness is a stronger condition than the non-existence of arbitrage, as such we'd expect stricter requirements to be satisfied. Theorem 22 stated that the existence of a martingale measure $\mathbb{Q}$ (or equivalently existence of a vector process $\mathbf{u}$ satisfying (8.2)) is sufficient to ensure the market is arbitrage-free. Market completeness requires that the measure $\mathbb{Q}$ (and hence $\mathbf{u}$) to be unique.

Theorem 23. *The market is complete if and only if there is a unique martingale measure $\mathbb{Q}$ for the discounted stock price process.*

There is a 1 : 1 correspondence between $\mathbb{Q}$ and $\mathbf{u}$ (according to relation (8.1)), and so $\mathbb{Q}$ will be unique if and only if $\mathbf{u}$ is the unique solution in $\mathbb{R}^m$ to (8.2). It follows that the necessary and sufficient requirement for a market to be complete is that $\mathrm{rank}(\sigma_t) = m$ almost always.

Why does a unique martingale measure imply market completeness? Assume $\mathbb{Q}$ exists and is unique, in this case $\mathbf{u}$ must be uniquely defined by (8.2), therefore $\mathrm{rank}(\sigma_t) = m$. The market is complete if a process ϕ can be found satisfying (8.4) and (8.5). Discounting Equation (8.5) by a factor of B_T^{-1} and applying the change of variables $\tilde{S}_t = B_t^{-1} S_t$, $\tilde{W}_t = \int_0^t \mathbf{u}_s ds + W_t$, we have

$$
B_T^{-1} C_T = V_0 + \int_0^T B_t^{-1} \sum_{i=1}^d \phi_t^{(i)} \sigma_t^{(i)} d\tilde{W}_t.
$$

But by the martingale representation theorem (Theorem 21), we must be able to find a unique $\theta_t = \left(\theta_t^{(1)}, \theta_t^{(2)}, \ldots, \theta_t^{(m)} \right)$ such that

$$
B_T^{-1} C_T = C_0 + \int_0^T B_t^{-1} \theta_t d\tilde{W}_t,
$$

thus $C_0 = V_0$ and

$$\int_0^T B_t^{-1} \sum_{j=1}^m \sum_{i=1}^d \phi_t^{(i)} \sigma_t^{(ij)} d\tilde{W}_t^{(j)} = \int_0^T B_T^{-1} \sum_{j=1}^m \theta_t^{(j)} d\tilde{W}_t^{(j)}.$$

Since each component $\tilde{W}_t^{(j)}$ of $\tilde{W}_t$ wanders independently, the only way that this can occur is if

$$\sum_{i=1}^d \phi_t^{(i)} \sigma_t^{(ij)} = \theta_t^{(j)}, \quad \forall t \geq 0, j = 1, \ldots, m. \tag{8.6}$$

By assumption σ_t is of rank m which means ϕ_t is uniquely defined by this system. Since C was arbitrary, any claim may be replicated in this way. Moreover we have the freedom to choose $\psi = (\psi_t; t \geq t)$ such that the self-financing condition (8.4) is satisfied.

Why is a unique martingale measure required for market completeness? We shall argue that unless $\text{rank}(\sigma_t) = m$ not every possible claim C will be attainable. This is easy to see since attainability requires that the system (8.6), namely

$$\phi_t \sigma_t = \theta_t, \quad \forall t \geq 0,$$

can be solved (for ϕ). Thus θ_t must be part of the linear span of the rows of σ_t. And since θ could correspond to any claim, this linear span must be all of $\mathbb{R}^m$.

Example 35. *Consider a market model with $B_t = 1, \forall t$, and three risky assets*

$$\begin{aligned} dS_t^{(1)} &= 2dt + dW_t^{(1)}, \\ dS_t^{(2)} &= dt + 2dW_t^{(2)}, \\ dS_t^{(3)} &= dt + dW_t^{(1)} - 2dW_t^{(2)}, \end{aligned}$$

Then $r = 0$ and Equation (8.2) gets the form

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_t^{(1)} \\ u_t^{(2)} \end{pmatrix},$$

which has the unique solution $\left(u^{(1)}, u^{(2)} \right) = \left(2, \frac{1}{2} \right)$. Therefore the market is complete.

Example 36. *Consider a market model with $B_t = 1, \forall t$, and three risky assets driven by 3-dimensional Brownian motion (m = 3).*

$$
\begin{aligned}
dS_t^{(1)} &= dt + dW_t^{(1)}, \\
dS_t^{(2)} &= 2dt + dW_t^{(2)} + dW_t^{(3)}, \\
dS_t^{(3)} &= 3dt + dW_t^{(1)} + dW_t^{(2)} + dW_t^{(3)}.
\end{aligned}
$$

For this model (8.2) translates as

$$
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{pmatrix},
$$

which has a solution space (e.g. $\mathbf{u} = (1, 1, 1)^t$) but since σ is only of rank 2, there is no unique solution. The model is arbitrage-free but not complete.

Indeed the $\mathcal{F}_T$-measurable claim with payoff $C_T = W_T^{(2)}$ cannot be replicated by a portfolio containing quantities of $S_t^{(1)}$, $S_t^{(2)}$, $S_t^{(3)}$, and B_t. To see why, let's seek such a replicating portfolio.

We are looking for a $\phi_t = \left(\psi_t, \phi_t^{(1)}, \phi_t^{(2)}, \phi_t^{(3)} \right)$ such that

$$
\begin{aligned}
B_T^{-1} W_T^{(2)} &= B_T^{-1} V_0 + \int_0^t B_t^{-1} \phi_t^{(1)} dW_t^{(1)} + \int_0^t B_t^{-1} \phi_t^{(2)} \left(dW_t^{(2)} + dW_t^{(3)} \right) \\
&\quad + \int_0^t B_t^{-1} \phi_t^{(3)} \left(dW_t^{(1)} + dW_t^{(2)} + dW_t^{(3)} \right), \\
&= B_T^{-1} V_0 + \int_0^t B_t^{-1} \left(\phi_t^{(1)} + \phi_t^{(2)} \right) dW_t^{(1)} + \int_0^t B_t^{-1} \left(\phi_t^{(2)} + \phi_t^{(3)} \right) dW_t^{(2)} \\
&\quad + \int_0^t B_t^{-1} \left(\phi_t^{(2)} + \phi_t^{(3)} \right) dW_t^{(3)}.
\end{aligned}
$$

By the martingale representation theorem, there is a unique $\theta_t = \left(\theta_t^{(1)}, \theta_t^{(2)}, \theta_t^{(3)} \right)$ such that

$$
B_T W_T^{(2)} = C_0 + \int_0^T B_t^{-1} \theta_t^{(1)} dW_t^{(1)} + \int_0^T B_t^{-1} \theta_t^{(2)} dW_t^{(2)} + \int_0^T B_t^{-1} \theta_t^{(3)} dW_t^{(3)},
$$

and by inspection $\theta = (0, 1, 0)$. The portfolio we are seeking must therefore satisfy

$$
\begin{aligned}
\phi_t^{(1)} + \phi_t^{(2)} &= 0, \\
\phi_t^{(2)} + \phi_t^{(3)} &= 1, \\
\phi_t^{(2)} + \phi_t^{(3)} &= 0.
\end{aligned}
$$

It is easy to see that no such ϕ exists.

8.3 Option pricing

If a given stock market model is recognisable as part of the semimartingale class, then solutions to economic problems such as option pricing and portfolio optimisation can be built on very solid platforms.

The following result summarises some of the discussion of the previous two sections.

Corollary 24. *Let $S_t = \left(B_t, S_t^{(1)}, \ldots, S_t^{(d)}\right)$ be a complete market. Suppose $\mathbf{u}$ satisfies (8.2), define $\mathbb{Q}$ by (8.1), and suppose $\tilde{W}$ to be $\mathbb{Q}$-Brownian motion. Let C_T be the payoff function of a European style claim, then the time t $(t < T)$ price of the claim is given by*

$$
C_t = B_t \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1} C_T | \mathcal{F}_t\right).
$$

Moreover, the replicating portfolio $\phi_t = \left(\psi_t, \phi_t^{(1)}, \ldots, \phi_t^{(d))}\right)$ is the unique solution to

$$
\begin{aligned}
\sum_{i=1}^{d} \phi_t^{(i)} \sigma_t^{(ij)} &= \theta_t^{(j)}, \quad j = 1, \ldots, m, \\
\psi_t &= B_t^{-1}\left(C_t - \sum_{i=1}^{d} \phi_t^{(i)} S_t^{(i)}\right),
\end{aligned}
$$

where θ satisfies

$$
B_T^{-1} C_T = C_0 + \int_0^T B_t^{-1} \theta_t d\tilde{W}_t.
$$

Actually, this result may fail if certain technical conditions aren't satisfied (see Øksendal (1998) theorems 12.3.2 and 12.3.4).

Example 37 (The Black & Scholes model). *Suppose the market has just two securities, a bond $dB_t = r_t B_t dt$ and a risky stock $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$ ($\sigma_t > 0$), where W_t is 1-dimensional Brownian motion. For this model Equation (8.2) reads*

$$\mu_t S_t - r_t S_t = \sigma_t S_t u_t,$$

which has a solution $u_t = \sigma_t^{-1}(\mu_t - r_t)$.

Using Corollary 24 we can price any European claim with payoff C_T via the formula

$$C_t = B_t \mathbb{E}_{\mathbb{Q}} \left(B_T^{-1} C_T \right), \tag{8.7}$$

where $\mathbb{Q}$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{(\mu_s - r_s)}{\sigma_s} dW_s - \frac{1}{2} \int_0^T \frac{(\mu_s - r_s)^2}{\sigma_s^2} ds \right).$$

Example 38 (Black & Scholes price for a European call option). *Assume r, μ, and σ are all constant. Substituting $B_T^{-1} C_T = (B_T^{-1} S_T - B_T^{-1} K)^+$ into (8.7) yields Black-Scholes formula*

$$C_0 = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

$$\begin{aligned} where \quad d_1 &= \frac{1}{\sigma\sqrt{T}} \left[\log\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T \right], \\ and \quad d_2 &= \frac{1}{\sigma\sqrt{T}} \left[\log\left(\frac{S_0}{K}\right) + (r - \frac{1}{2}\sigma^2)T \right]. \end{aligned}$$

8.4 Exercises

1. If $\tilde{C}_t = B_t^{-1} C_t$ and $\tilde{S}_t = B_t^{-1} S_t$ are both $\mathbb{Q}$-martingales, and ϕ is a process chosen such that $d\tilde{C}_t = \phi_t d\tilde{S}_t$ for all t. Show that C is replicated by holding ϕ_t units of stocks and $\psi_t = B_t^{-1} (C_t - \phi_t S_t)$ bonds for all $t \le T$.

2. If $dS_t = \mu_t dt + \sum_{j=1}^m \sigma_t^{(j)} dW_t^{(j)}$, $dB_t = rB_t dt$, and $\tilde{W}_t^{(j)} = \int_0^t u_s^{(j)} ds + W_t^{(j)}$, show that

$$d\left(B_t^{-1}S_t\right) = B_t^{-1}\left(\mu_t - r_t S_t - \sum_{j=1}^m \sigma_t^{(j)} u_t^{(j)}\right)dt + B_t^{-1}\sum_{j=1}^m \sigma_t^{(j)} d\tilde{W}_t^{(j)}.$$

3. In each of the following,

   - determine whether or not arbitrage opportunities exist (assume $B_t = 1, \forall t$). If so, find one;

   - determine if the markets are complete. If not, find a claim which is not attainable.

   (a)

$$\begin{aligned}
dS_t^{(1)} &= 3dt + dW_t^{(1)} + dW_t^{(2)}, \\
dS_t^{(2)} &= -dt + dW_t^{(1)} - dW_t^{(2)}.
\end{aligned}$$

   (b)

$$\begin{aligned}
dS_t^{(1)} &= dt + dW_t^{(1)} + dW_t^{(2)} - dW_t^{(3)}, \\
dS_t^{(2)} &= 5dt - dW_t^{(1)} + dW_t^{(2)} + dW_t^{(3)}.
\end{aligned}$$

   (c)

$$\begin{aligned}
dS_t^{(1)} &= dt + dW_t^{(1)} + dW_t^{(2)}, \\
dS_t^{(2)} &= 2dt + dW_t^{(1)} - dW_t^{(2)}, \\
dS_t^{(3)} &= 3dt - dW_t^{(1)} + dW_t^{(2)},
\end{aligned}$$

   (d)

$$\begin{aligned}
dS_t^{(1)} &= dt + dW_t^{(1)} + dW_t^{(2)}, \\
dS_t^{(2)} &= 2dt + dW_t^{(1)} - dW_t^{(2)}, \\
dS_t^{(3)} &= -2dt - dW_t^{(1)} + dW_t^{(2)},
\end{aligned}$$

4. If

$$C_T = V_0 + \int_0^T \psi_t dB_t + \sum_{i=1}^d \int_0^T \phi_t^{(i)} dS_t^{(i)},$$

show that

$$B_T^{-1} C_T = V_0 + \int_0^T B_t^{-1} \sum_{i=1}^d \phi_t^{(i)} \sigma_t^{(i)} d\tilde{W}_t,$$

where $\tilde{S}_t = B_t^{-1} S_t$, $\tilde{W}_t = \int_0^t \mathbf{u}_s ds + W_t$, and $\mathbf{u}$ satisfies $\mu_t - r_t S_t = \sigma_t \mathbf{u}_t$.

5. Evaluate (8.7) for the case described in Example 38.