Chapter 5

Brownian Motion

Several aspects of the binomial asset pricing models of Chapter 3 fail to capture financial markets' real features. Perhaps the most crucial flaw in these models is that prices are restricted to lie on a lattice. The log-normal model alleviates some of the problems but is less general than we would like; it lacks generality in the sense that only processes which have an exponentially increasing trend with noise can be catered for. Interest rates, for example, don't necessarily have this shape. Neither do energy prices follow a shape which agrees with the log-normal distribution (much to the dismay of several researchers in the area).

The modern theory of Brownian driven continuous-time models dates back to the 1950's and '60's, when researchers seemingly rediscovered the thesis of Bachelier (1900). Apparently introduced before its time[1], his idea of using Brownian motion as the driving process behind a stock market model is widely used today.

Brownian motion is named after the Scottish botanist Robert Brown who described the motion of a pollen particle suspended in fluid in 1828. Brown observed that the particle continually moved randomly on the surface. Einstein suggested in 1905 that the movements were due to collisions between the liquid's molecules and the relatively light pollen.

Applications of Brownian motion are not limited to the study of pollen suspended in liquid. It has successfully been used to describe thermal noise in electrical circuits,

[1]Bachelier's thesis was rejected (Korn 1997).

limiting behaviour of queueing networks under heavy traffic, population dynamics in biological systems, and of course it has been used in modelling various economic processes.

Any book which deals with stochastic calculus will also provide a thorough treatment of Brownian motion. Good introductions may be sought from Øksendal (1998), Klebaner (1998), Chung & Williams (1990), Durrett (1996), Protter (1992), or Mikosch (1998), for example. Alternatively one may like to look at a specialised text such as Karatzas & Shreve (1991),or Revuz & Yor (1990).

A stochastic process $W = (W_t, t \in [0, \infty))$ is called *(standard) Brownian motion* or a *Wiener process* if it starts at zero ($W_0 = 0$) and the following conditions are satisfied:

Gaussian process for each $s \geq 0$ and $t > 0$ the random variable $W_{t+s} - W_s$ has the normal distribution with mean zero and variance t;

independent increments for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ the random variables $\{W_{t_r} - W_{t_{r-1}}\}$ are independent;

continuity with probability one, $t \mapsto W_t$ is continuous.

Some of the properties of Brownian motion which are already apparent are:

- $W_t - W_s$, $s < t$ has the same distribution as W_{t-s}, namely $N(0, t-s)$;
- $W_{t\,s}$, $s > 0$ has the same distribution as $\sqrt{t}\,W_s$, namely $N(0, t\,s)$;
- $\mathbb{E}(W_t) = 0, \quad \forall t > 0$;
- $Cov(W_t, W_s) = \mathbb{E}(W_t\,W_s) - \mathbb{E}(W_t)\,\mathbb{E}(W_s) = s$,
  (rewrite $W_t\,W_s$ as $(W_t - W_s)W_s + W_s^2$ and use the independence of $W_t - W_s$ and W_s).

The easiest way to think about Brownian motion is as an infinitesimal random walk. This is not, however, the easiest way to construct the process. Therefore we describe the process as a random walk and then construct it differently.

5.1 Frogger

Frogger is a computer game which reached its height of popularity in the mid to late 80's. One basic version of Frogger consists of a multi-lane highway with traffic and a small frog. The player's objective is to simply guide the frog out of the paths of oncoming cars and trucks. In order to do this the frog must jump from its current position up or down a single lane at each of a number of equally spaced times. We can regard the direction of each jump as being random because the stream of cars is assumed random. Therefore the position of the frog after the nth jump is the result of a random walk.

The game starts off relatively easy with only light traffic all moving slowly. If the frog survives for some period, say t, then the game restarts at the next level of difficulty - more cars, moving faster. To compensate for the increasing numbers of cars, the frequency of jumps the frog makes also increases. The game continues: after each period of length t the frog needs to make a greater number of jumps if it is to survive.

As the number of steps per period, n, increases so does the potential distance the frog may move from its starting position. To keep things on the screen the program scales the size of each lane to be $\sqrt{\frac{t}{n}}$. The scaling parameter $\sqrt{\frac{t}{n}}$ is chosen because it is the only scaling which leads to a sensible limit as $n \to \infty$. If the lane sizes are bigger than $\sqrt{\frac{t}{n}}$ the random walker may reach infinity in a finite time, whereas if the step sizes are smaller than $\sqrt{\frac{t}{n}}$ then the limiting process will have no variation at all.

We have set up a sequence of random walks indexed by n, $X^{(n)} = \left(X_j^{(n)}; j = 0, \ldots, n\right)$

$$\begin{aligned} X_j^{(n)} &= X_0 + \sum_{k=1}^{j} \xi(k), \quad \text{where} \\ \xi(k) &= \begin{cases} \sqrt{\frac{t}{n}}, & \text{w.p. } \frac{1}{2}, \\ -\sqrt{\frac{t}{n}}, & \text{w.p. } \frac{1}{2}, \end{cases} \end{aligned}$$

and now the obvious thing to do is apply the central limit theorem and see that the random walker $X^{(n)}$ approaches (as $n \to \infty$) a random process W_t with the properties of Brownian motion as outlined above. This doesn't prove that such a process exists.

The problem with the random walk construction is that the sample paths are not continuous. No matter how large the number of steps n there will always be a discontinuity between sequential positions $\left(j\frac{t}{n}, X_j^{(n)}\right)$, $\left((j+1)\frac{t}{n}, X_{j+1}^{(n)}\right)$, whereas Brownian motion is continuous.

More formally, consider the σ-field generated by the finite dimensional sets

$$\mathcal{F}_n = \{\omega : \omega(t_i) \in A_i \quad \text{for} \quad 1 \leq i \leq n\}.$$

Each A_i corresponds to the set of possible positions this walk could be at after i steps. So $\mathcal{F}_n$ may be regarded as the collection of all possible sample paths of the random walk with n steps. Our problem is that we are trying to construct a continuous process with sample paths in the set

$$C = \{\omega : t \mapsto \omega(t) \quad \text{is continuous}\},$$

but for every n, $C \notin \mathcal{F}_n$. That is to say, C is not an $\mathcal{F}_n$-measurable set.

Brownian motion may be constructed using the random walk approach although it becomes very technical. There are other rigorous methods of proving Brownian motion exists. Kolmogorov's extension theorem provides one of the neater proofs, as outlined by Øksendal (1998). The approach we shall take is not as neat but avoids the measure theoretic aspects of the extension theorem method.

5.2 Lévy's Construction

History has proven that showing existence of Brownian motion is not an easy task. Bachelier (1900) is credited as being the first to attempt a quantitative analysis, but his construction was later shown to be erroneous (Mikosch 1998). The first rigorous construction was given by Wiener in 1931, for this reason this process is also frequently called a *Wiener process*. Wiener's proof was later modified by Lévy. Lévy's proof is the one we shall sketch out here.

Lévy's construction proceeds as follows. Define inductively a sequence of processes $X^{(n)} = (X^{(n)}(t); t \geq 0)$. Without loss of generality we take the range of t to be $[0, 1]$.

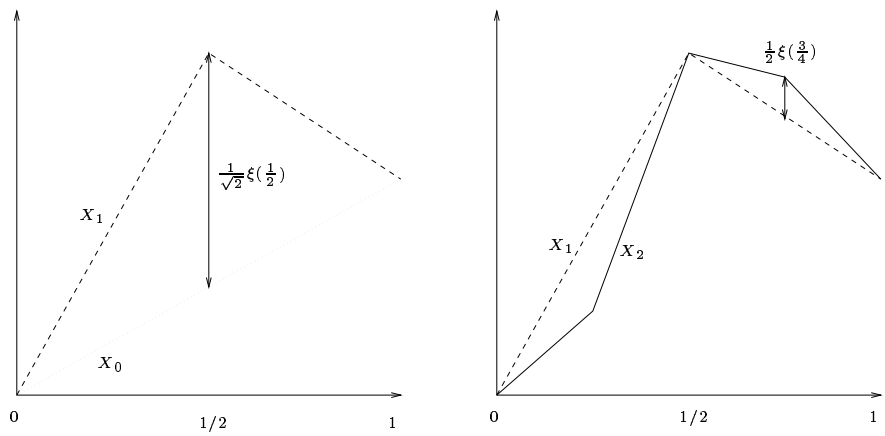

Figure 5.1: Lévy's construction of Brownian motion.

We need a countable number of independent standard normal random variables. Index these by the dyadic rationals of $[0,1]$ ($\{k2^{-n} : k, n \geq 0\}$). In other words, we have for each n a sequence of $N(0,1)$ random variables labelled

$$\{\xi(k2^{-n})\}_{k=1}^{2^n}.$$

Our induction begins with $X^{(0)}(t) = t\,\xi(1)$, thus $X^{(0)}$ is a linear function on $[0,1]$. Next define $X^{(1)}(t)$ to be equal to $X^{(0)}(t)$ for $t = 0, 1$, but passes via a new point $X^{(0)}(\frac{1}{2}) + \frac{1}{\sqrt{2}}\xi(\frac{1}{2})$. Formally

$$X^{(1)}(t) = \begin{cases} t\left(X^{(0)}(\frac{1}{2}) + \frac{1}{\sqrt{2}}\xi(\frac{1}{2})\right), & 0 \leq t < \frac{1}{2}; \\ X^{(0)}(\frac{1}{2}) + \frac{1}{\sqrt{2}}\xi(\frac{1}{2}) + 2(t - \frac{1}{2})\left(\xi(1) - \left(X^{(0)}(\frac{1}{2}) + \frac{1}{\sqrt{2}}\xi(\frac{1}{2})\right)\right), & \frac{1}{2} \leq t < 1. \end{cases}$$

Effectively we are picking up the mid-point of $X^{(0)}(t)$ and shifting it causing $X^{(1)}(t)$ to be a bent version of $X^{(0)}(t)$. The new function $X^{(1)}(t)$ is linear on the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$. Similarly the nth function $X^{(n)}(t)$ is linear on the intervals $[(k-1)2^{-n}, k2^{-n})$, $k = 1, \ldots, 2^n$, and is thus determined by its values $X^{(n)}(k2^{-n})$, and $X^{(n)}(0) = 0$.

Now for the inductive step. We take

$$X^{(n+1)}(2k2^{-(n+1)}) = X^{(n)}(2k2^{-(n+1)}) = X^{(n)}(k2^{-n}).$$

That is to say, all the points which are corners in the previous process $X^{(n)}(t)$ are also fixed corners in the next process. Half way between each of those corners we bend the line again forming new corners according to the rule

$$X^{(n+1)}((2k-1)2^{-(n+1)}) = X^{(n)}((2k-1)2^{-(n+1)}) + 2^{-\frac{n}{2}}\xi\left(\frac{2k-1}{2^n}\right).$$

Figure 5.1 may be useful to help visualise this construction process.

To prove that Brownian motion exists we need to check the following:

Lemma 5. *Two conditions for existence of Brownian motion:*

1. *with probability* 1, $W_t = \lim_{n\to\infty} X^{(n)}(t)$ *exists for* $0 \le t \le 1$ *uniformly in* t*;*

2. *furthermore,* W_t *defined as this limit if it exists and zero otherwise satisfies the conditions of the definition for Brownian motion.*

Proof. We are aiming to show

$$\lim_{n\to\infty} \Pr\left[\text{ for some } k > n, \text{ and every } 0 \le t \le 1,\ |X^{(k)}(t) - X^{(n)}(t)| \ge \epsilon\right] = 0.$$

Our starting point is to consider

$$\begin{aligned}
\Pr\left[\max_t |X^{(n+1)}(t) - X^{(n)}(t)| \ge 2^{-n/4}\right] &= \Pr\left[\max_{1\le k\le 2^n} \xi\left((2k-1)2^{-(n+1)}\right) \ge 2^{n/4}\right], \\
&\le 2^n \Pr(\xi(1) \ge 2^{n/4}), \\
&\le 2^n \exp(2^{n/2}), \\
&< 2^{-n}.
\end{aligned}$$

The last step is justified since

$$\lim_{n\to\infty} \frac{2^{n/2}}{2n\log(2)} \to \infty,$$

therefore $\exp(-2^{n/2}) \to 0$ faster than $2^{-2n} \to 0$. From which we reason

$$\exists k > 0 : \exp(2^{n/2}) < 2^{-2n} \quad \forall n \ge k.$$

Now for any $m > n \geq k$

$$\Pr\left[\max_t |X^{(m)}(t) - X^{(n)}(t)| \geq 2^{-n/4}\right] = 1 - \Pr\left[\max_t |X^{(m)}(t) - X^{(n)}(t)| \leq 2^{-n/4}\right],$$

and

$$\begin{aligned}
\Pr\left[\max_t |X^{(m)}(t) - X^{(n)}(t)| \leq 2^{-n/4}\right] &\geq \Pr\left[\sum_{j=n}^{m-1} \max_t |X^{(j+1)}(t) - X^{(j)}(t)| \leq 2^{-n/4}\right], \\
&\geq \Pr\left[\cap_{j=n}^{m-1} \max_t |X^{(j+1)}(t) - X^{(j)}(t)| \leq 2^{-j/4}\right],
\end{aligned}$$

since $\sum_{j=n}^{m-1} 2^{-j/4} = 2^{-n/4}\left(\frac{1-2^{-(m-1-n)/4}}{1-2^{1/4}}\right) \leq 2^{-n/4}$. So

$$\begin{aligned}
\Pr\left[\max_t |X^{(m)}(t) - X^{(n)}(t)| \leq 2^{-n/4}\right] &\geq 1 - \Pr\left[\cup_{j=n}^{m-1} \max_t |X^{(j+1)}(t) - X^{(j)}(t)| \geq 2^{-j/4}\right], \\
&\geq 1 - \sum_{j=n}^{m-1} \Pr\left[\max_t |X^{(j+1)}(t) - X^{(j)}(t)| \geq 2^{-j/4}\right], \\
&\geq 1 - \sum_{j=n}^{m-1} 2^{-j}, \\
&\geq 1 - 2^{-n+1}.
\end{aligned}$$

Finally we have that

$$\Pr\left[\max_t |X^{(m)}(t) - X^{(n)}(t)| \geq 2^{-n/4}\right] \leq 2^{-n+1}, \quad \text{for } m > n \geq k.$$

Next recall a property from probability theory regarding the limit of an increasing sequence of events. If $\{A_n\}$ is an increasing sequence of events, that is $A_1 \subseteq A_2 \subseteq \ldots$, then $\lim_{n\to\infty} A_n = \cup_{i=1}^{\infty} A_i$, and satisfies $\Pr(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \Pr(A_n)$. Grimmett & Stirzaker (1997) contains all sorts of useful probability relations.

Now $\{\max_t |X^{(m)}(t) - X^{(n)}(t)| \geq 2^{-n/4}\}_{m=n}^{\infty}$ is an increasing sequence of events since the maximum can only increase by addition of a new vertex, so

$$\Pr\left[\max_t |X^{(m)}(t) - X^{(n)}(t)| \geq 2^{-n/4} \text{ for some } m > n\right] \leq 2^{-n+1}.$$

In particular,

$$\lim_{n\to\infty} \Pr\left[\text{for some } m > n \text{ and all } 0 < t < 1, |X^{(m)}(t) - X^{(n)}(t)| \geq \epsilon\right] = 0,$$

which completes part 1 of the lemma.
Part 2 is easy to check: $X^{(n)}(t)$ satisfies all the properties (except continuity) for $t \in \{k2^{-n}\}_{k=1}^{2^n}$. This is also true of the processes $X^{(m)}$ for $m > n$ since the values at these times don't change. Therefore it must also be true for W on $\cup_{n=1}^{\infty}\{k2^{-n}\}_{k=1}^{2^n}$.

Continuity of the limiting process is established by observing that the uniform limit of a sequence of continuous functions is continuous.

Finally, since each $t \in [0,1]$ may be approximated arbitrarily closely by a sum of the form $\sum_{j=1}^{\infty} a_j 2^{-j}$ where a_j is either zero or one, all the defining properties of Brownian motion must hold for any sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1$ from within $\cup_{n=1}^{\infty}\{k2^{-n}\}_{k=1}^{2^n}$. □

Brownian motion is difficult to imagine and impossible to draw, when a text book displays a sample path of "Brownian Motion" they are really graphing an infinitesimal random walk.

- Brownian motion is a fractal, one can zoom in on a sample path as many times as you like but the jaggedness never smooths out;

- Brownian motion is nowhere differentiable. No matter how small a segment you choose, you will never (with probability zero) find a Brownian motion segment which is a straight line, even though we constructed it from such pieces;

- Brownian motion sample paths do not have bounded variation on any finite interval $[0,T]$, that is

$$\sup_{\tau} \sum_{i=1}^{n} |W_{t_i}(\omega) - W_{t_{i-1}}(\omega)| = \infty,$$

  where the supremum is taken over all possible partitions $\tau : 0 = t_0 < \cdots < t_n = T$. In other words, if we wanted to lay a piece of string down on a Brownian motion sample path in the interval $[0,T]$, we would need an infinitely long string regardless of how small T;

- physically speaking, if a particle is to perform Brownian motion then it must have zero mass, otherwise how can it follow a non-smooth path?

5.3 Brownian driven stock models

When viewed on a small time scale, stock prices seem very similar to Brownian motion. They are both jagged processes and this jaggedness never seems to smooth under magnification. The stock price process seems to be forever changing direction, just like Brownian motion.

Globally however, the similarity is less striking. Figure 5.2 compares a relatively recent data set of Commonwealth bank share prices to standard Brownian motion. The two major differences between these series are:

- the stock price process appears to become more volatile as time passes. This change in volatility is in the sense that the instantaneous jumps are getting larger. Brownian motion, on the other hand, has a variance which is constant in time;

- standard Brownian motion may go negative, while the stock market we are considering has strictly positive prices.

Brownian motion, as it is defined, is not a good model for the CBA stock price, but that's not to say we should abandon the idea. Our aim is to mould Brownian motion to have a shape similar to the one we are modelling. Once we have the shape, we can perform a probabilistic analysis on the model and form estimates of the quantities we are interested in: things like "what's the likelihood of a stock losing more than 10% of it's value in a day", or "how much should we charge for this option contract?"

We have established that standard Brownian motion does not make a good model for the stock price process which we are interested. Firstly, the stock price process may have a long term trend for growth or decay. Brownian motion wanders around its mean, zero. As a partial solution we can add a drift term and fit the model

$$S_t = \mu\, t + W_t.$$

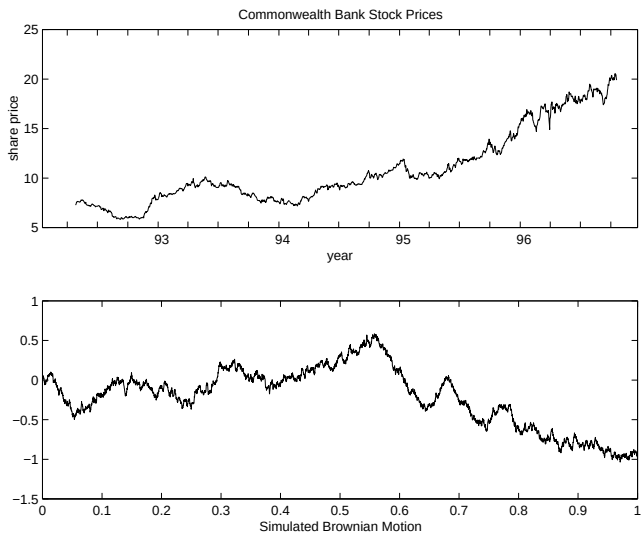

Figure 5.2: CBA stock prices and simulated (standard) Brownian motion

The model may be too noisy or not noisy enough, this can be rectified by incorporating a scaling factor

$$S_t = \mu\, t + \sigma\, W_t.$$

It is not difficult to check that the new process is a transformation of Brownian motion with mean $\mu\, t$ and variance $\sigma^2\, t$. The new process is still a Gaussian process, but with a shifted mean and scaled variance. We call this process *Brownian motion with linear drift.*

Now seems like a good time to check how closely our model reflects reality. Figure 5.3 gives an indication of how our process is likely to look. The graph clearly illustrates the effect of adding a drift and scaling the variance to the original Brownian motion. All-in-all we have a better model than the previous, but it's not glitch-free; the process can still go negative.

In their landmark papers Black & Scholes (1973) and Merton (1973) suggested another transformation as a model for prices, *Geometric Brownian motion.* This stochastic process is quite simply our Brownian motion with drift process passed to

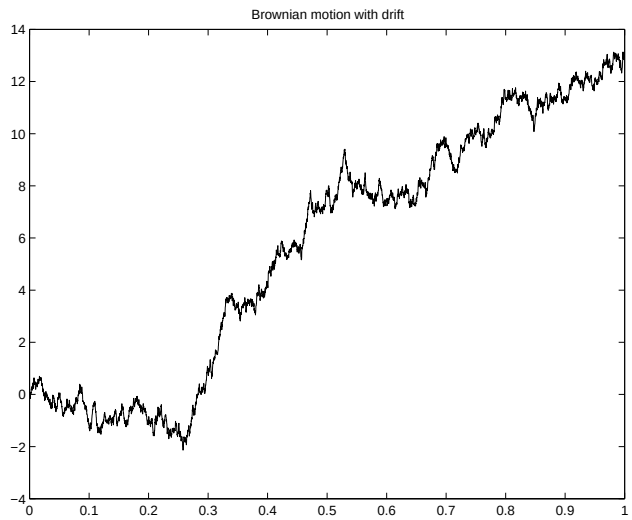

Figure 5.3: Simulated sample path of Brownian motion with drift; $\mu = 10$, and $\sigma^2 = 5$.

the exponential function

$$S_t = \exp(\mu\, t + \sigma\, W_t).$$

Figure 5.4 shows just how impressively similar this model is to reality. Note in particular the increasing variance. If one was to take greater care in selecting parameters μ and σ there is no doubt that this simple model could be put to practice.

5.4 Exercises

The Matlab files *brownian.m*, and *geombrownian.m* available at

`www.maths.uq.edu.au/~mrt/ms479/`

may help with these exercises.

1. Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion. Which of the following are also Brownian motion? Justify your answers.

   (a) $\{-W_t\}_{t\geq 0}$,

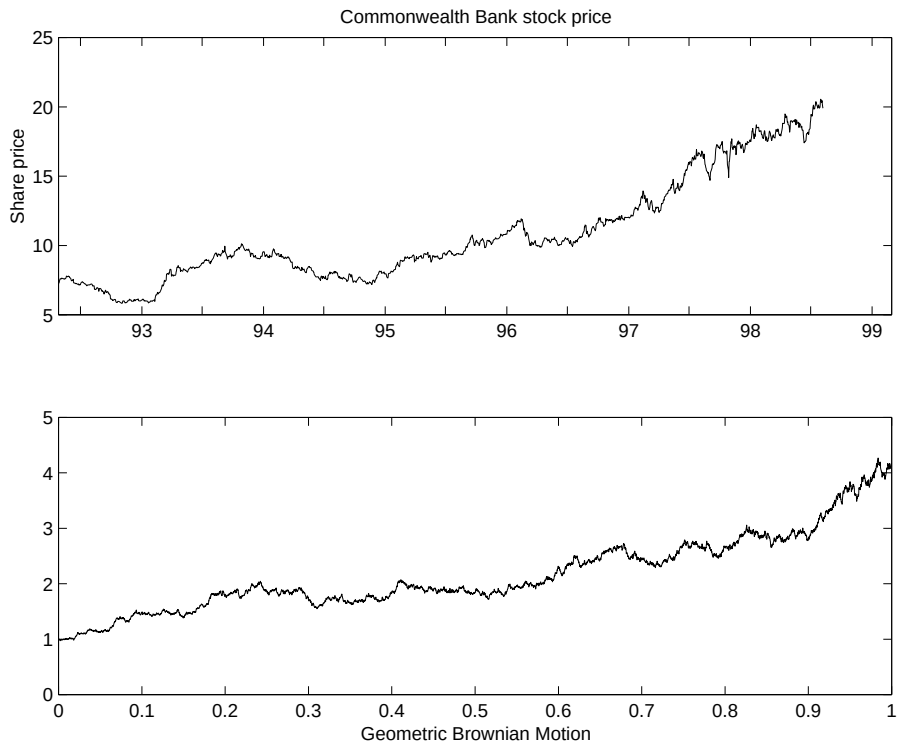

Figure 5.4: Commonwealth Bank stock prices and Simulated sample path of Brownian motion with drift; $\mu = 1$, and $\sigma^2 = .5$.

(b) $\{cW_{t/c^2}\}_{t\geq 0}$,

(c) $\{\sqrt{t}W_1\}_{t\geq 0}$,

(d) $\{W_{2t} - W_t\}_{t\geq 0}$.

2. Each pane in Figure 5.5 matches one of the equations:

$$S_t = e^{0.5\,t+W_t}, \tag{5.1}$$

$$S_t = e^{t+0.5\,W_t}, \tag{5.2}$$

$$S_t = 1 + 0.5\,t + W_t, \tag{5.3}$$

$$S_t = 1 + t + 0.5\,W_t. \tag{5.4}$$

Which matches which? What are the characteristics which allow you to tell them apart?

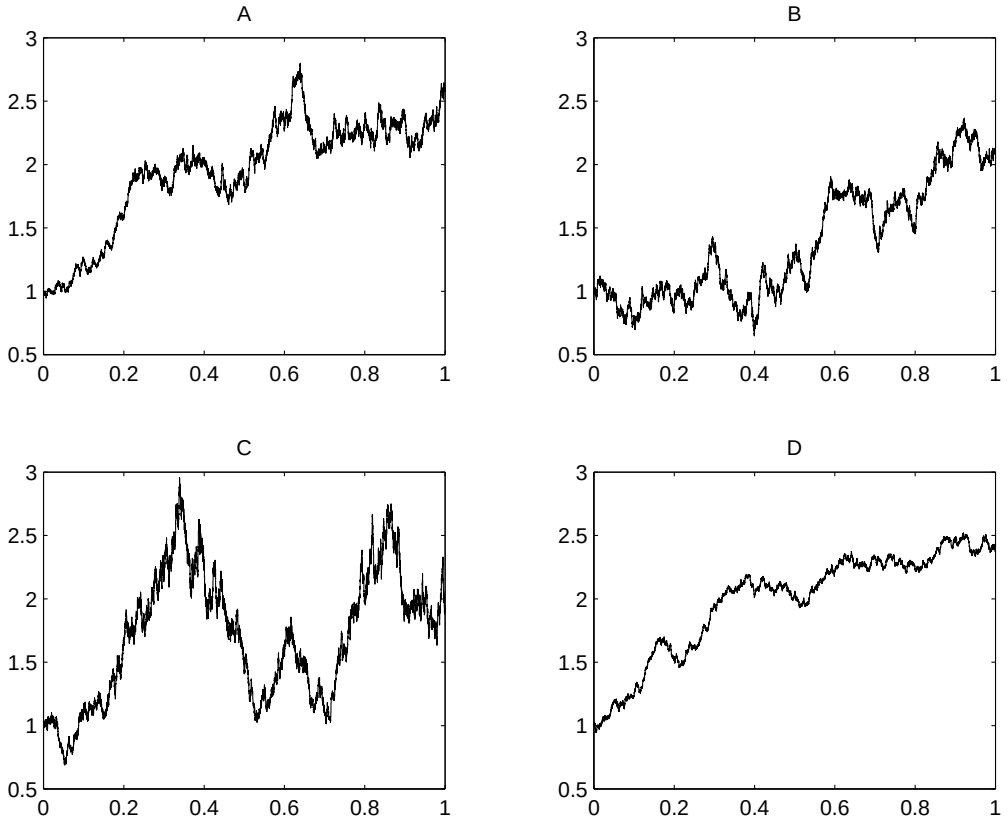

Figure 5.5: