Chapter 1

Introduction to financial derivatives

For the most part, we are concerned with evaluating the fair price of a variety financial contracts. The terms *financial derivatives* and *contingent claims* (and rearrangements of these) apply to the general class of contracts we will be dealing with. These include options, forwards, and futures. They are purely financial instruments designed to minimise the inherent risk an investor faces when trading stock. They are also be used for speculative betting. In this introductory section we shall take a detailed look at the simplest option type, the European options, before having a glancing look at some more exotic options.

1.1   European options

There are two main uses for options; speculation and hedging. A speculator will invest in a contract with a view to making a profit. Hedging, on the other hand, aims to minimise risk. For a hedger, options are like insurance policies, they may be used to fix upper price bounds for goods to be purchased, or to provide lower bounds for assets currently held.

The two standard European option contracts are call options and put options. A *European call option* gives the holder the right, but not the obligation, to buy

stock on a certain date T, for a specific price K. The date is sometimes called the *expiration date, exercise date*, or *time of maturity*, and the usual term for the specified price is the *strike*. A put is the same as a call with "buy" replaced with "sell".

Denote the price of the underlying stock as $S = (S_t, 0 \le t \le T)$. European options are characterised by the fact that their payoff function only depends on the stock price at maturity S_T.

If you hold a call option, then you can buy, if you wish, one share of stock at time T for K dollars. There are two possibilities. If at time T, $S_T < K$ then you will not exercise the option as you can buy the stock cheaper in the open market. In this case the option is worthless. In contrast, if $S_T > K$ then you can buy one share of stock for K and sell it immediately for S_T and make a profit of $S_T - K$. Thus a European call option has the value at time T of

$$C(S_T, T) = (S_T - K)^+ := \max(0, S_T - K).$$

By a similar argument the payoff of a European put option at expiry is

$$P(S_T, T) = (K - S_T)^+ := \max(0, K - S_T).$$

In later sections shall seek a value for the fair price of an option for times prior to expiry. In particular it will be useful to calculate the initial price (at time 0) of an option. This is the amount of money exchanged for the option contract T time units before maturity.

The notation $C(S, t)$ emphasises the dependence of the time-t price of an option on both the stock price process S and time t. When it's convenient we shall often abbreviate this to C_t.

Example 1. *Suppose a speculator suspects the price of oil will rise rapidly during the next three months (trouble in the Gulf). He outlays C_0 dollars for a three month European call option with strike price K. The price of oil fluctuates randomly during the quarter starting at S_0 and finishing at S_T. If at time T, $S_T > K$ (the option is in the money) then he exercises the option and makes a profit of $S_T - K - C_0$. Conversely, if $S_T < K$ the option is discarded and the speculator makes a loss of C_0.*

Call option profit
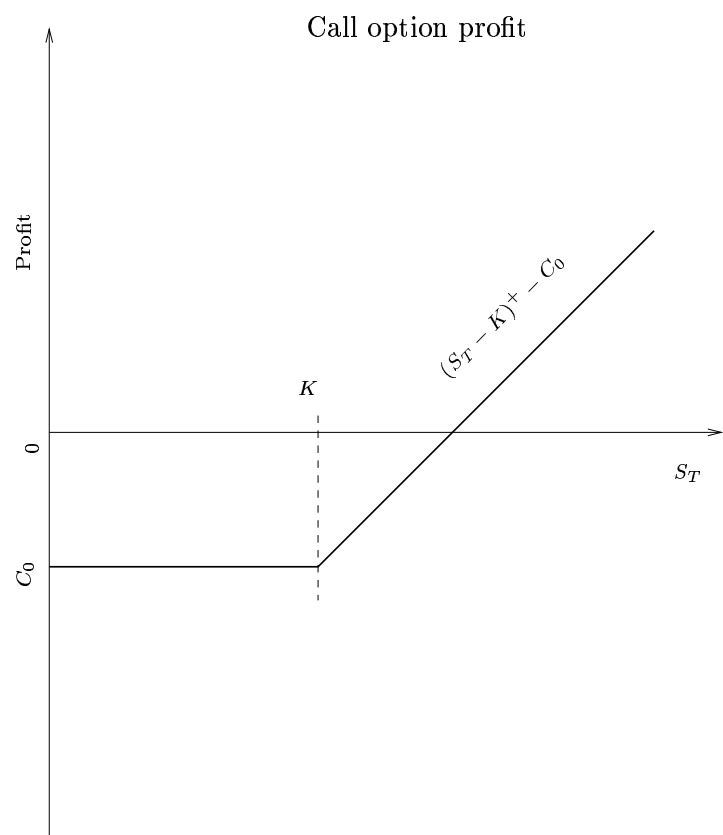
Profit

K

$(S_T - K)^+ - C_0$

0

S_T

C_0

Figure 1.1: Call option profit vs stock price at maturity

Example 2. *A large woolshed operation requires security on the value of its wool. Anticipated changes in government policy are expected to open the local market to foreign competition within the next year. And so a hedger is asked to ensure a minimum price for the wool. She purchases a twelve month put option with a strike price of K for P_0 dollars. At maturity, if the wool price, S_T, has fallen below the strike the woolshed still receives K dollars for the wool and the payoff from the put is $K - S_T$. If the stock price doesn't fall as expected and we find $S_T > K$ the put is discarded and the initial purchase P_0 is lost.*

For options with payoff functions that depend solely on the stock price at maturity it is meaningful to plot the payoff or profit against stock price. In the finance literature these graphs have the sensible title of payoff diagrams (resp. profit diagrams). The profit diagram for a European call option is given in Figure 1.1. The payoff diagram may be obtained by shifting the S_T-axis down C_0 units; effectively plotting C_T against S_T.

A major goal for researchers in this area is to be able to determine the rational price of an option for any stock price and any time. This problem has already been solved for many options. For the European call, the solution $C(x, t)$; $x \geq 0$, $0 \leq t \leq T$ satisfies a partial differential equation with boundary conditions $C(x, T)$; $x \geq 0$ and $C(0, t)$; $0 \leq t \leq T$. The function $C(S_T, T)$ is plotted in a payoff diagram for varying S_T. Figure 1.2 is a graph of the solution to the European call pricing problem. Notice the resemblance of the *time to maturity* $= 0$ plane to Figure 1.1, here the strike price is about $35.

Actually, there is a contract which has a simpler payoff function than that of the European options; the forward contract. When a forward contract is entered into, a trade of stock is agreed to take place at a future date for a price specified now. It costs nothing to enter into a forward contract, but because of this there is only one fair value for the strike price. Given the market environment, each forward contract has a unique strike price. This is in contrast to option contracts where there is freedom in choosing a strike and the initial price is affected by the choice. Figure 1.3 is the profit diagram for a forward, in this case the payoff diagram is the same.
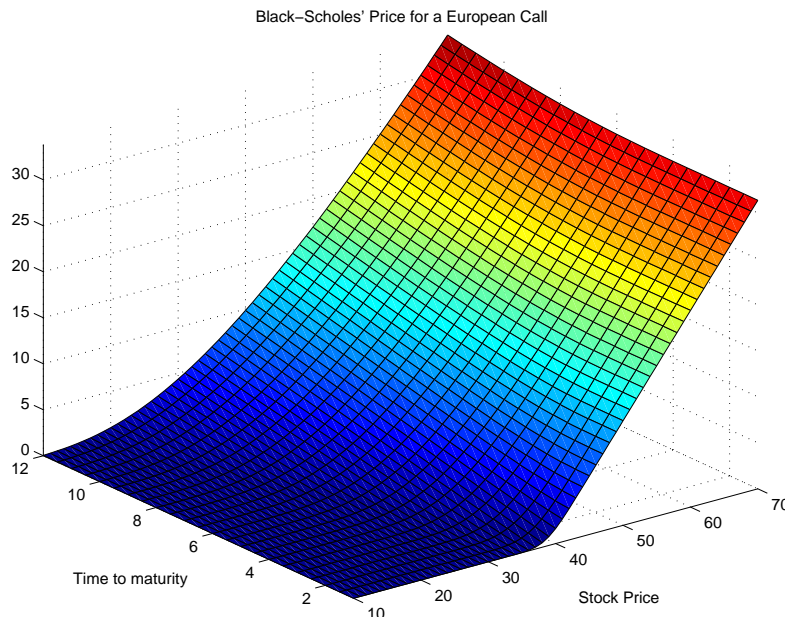
Black–Scholes' Price for a European Call

Figure 1.2: Call option value against time to maturity and stock price

1.1.1   Option strategies

By combining a variety of options into a single portfolio it is possible to tailor the payoff function. One example of an option strategy's payoff is given in Figure 1.4. This particular strategy has a payoff which can be described in three parts. If the stock price at maturity is less than K_1, then the payoff is the maximum possible $K_2 - K_1$. For values between K_1 and K_2 the payoff is $K_2 - S_T$. And if $S_T > K_2$ the portfolio is worth zero dollars.

One way of constructing such a payoff is by buying a put option with strike price K_2 (option A) and selling a put, having the same expiry date, with strike price $K_1 < K_2$ (option B). The three cases correspond to the events:

1. $K_2 < S_T$: both options are out of the money and are not exercised;

2. $K_1 < S_T \leq K_2$: option A is in the money, receive $K_2 - S_T$;

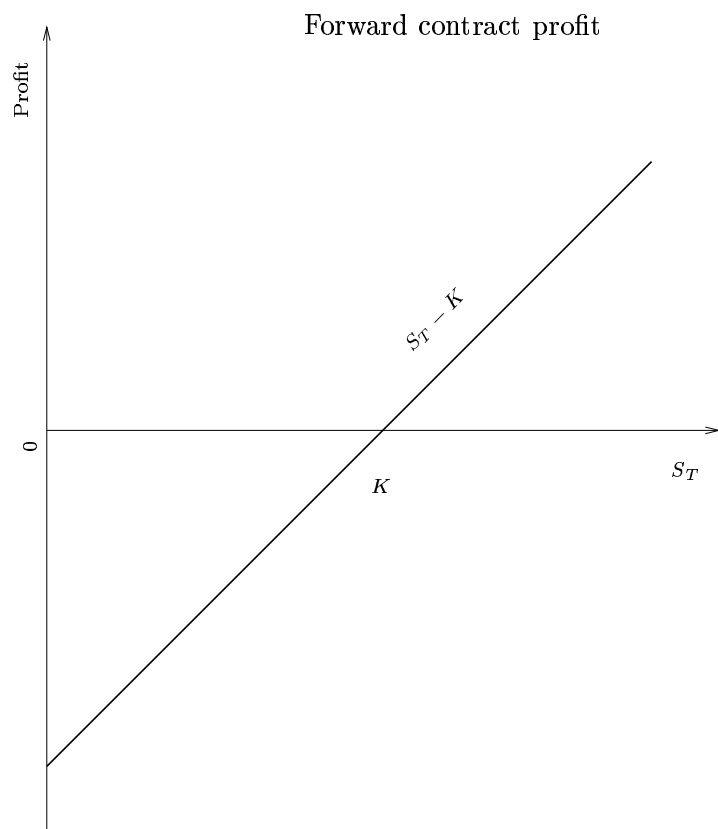3. $S_T \leq K_1$: both options are in the money, receive $K_2 - S_T$ from option A, pay

Figure 1.3: Forward contract profit vs stock price at maturity

Bear spread payoff

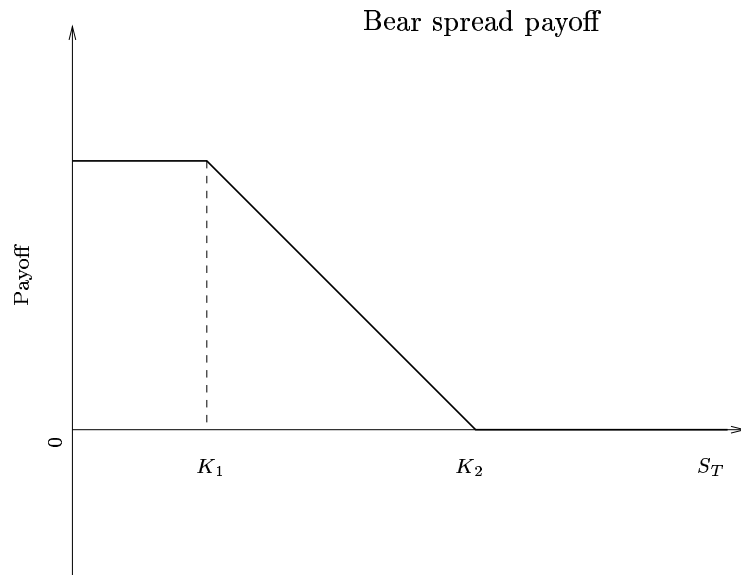
Figure 1.4: Payoff diagram for a bear spread

$K_1 - S_T$ to cover the obligation of option B. The total cash flow is $K_2 - K_1$.

This strategy is called a *bear spread* because it benefits from a bear (falling) market. The converse is a *bull spread*.

Example 3. *A new airline headed by a foreign entrepreneur has just announced it will be starting operations within the next year. It is expected that the added competition will cause the share value of a rival airline to fall. With this in mind an investor purchases a put option with strike price $36 and, to offset the expense, sells another put with strike $27. In doing so, the maximum loss he could incur is the cost to set up the portfolio; the difference between the initial put option values. The maximum payoff is limited to $9.*

1.1.2 Put-call parity

The value of a European put option may be expressed in terms of a call option with the same expiry date and strike price. This useful property is called the *put-call*

parity. A simple justification can also be achieved using payoff diagrams. This proof will be refined in Section 2.3, it uses arbitrage arguments developed in Section 2.1.

Proof. If the payoff diagram for a put option is subtracted from the payoff diagram for a call with the same strike K and maturity date, the payoff of Figure 1.3 is obtained. This is actually the graph of $S_T - K$, therefore $C_T - P_T = S_T - K$. This equality holds for all values of S_T.

Now if $C_T - P_T - S_T + K$ is worth zero regardless of S_T then its initial value must have also been worth zero (since no rational person will pay money for a portfolio which is guaranteed to be worthless on maturity). Thus

$$P_t = C_t - S_t + K, \qquad t \in [0, T]. \tag{1.1}$$

$\square$

1.2   A few exotic options

Option pricing provides a rich source of challenging mathematical problems for research. It has motivated research from (but not exclusively) control theorists, physicists, statisticians, operations researchers, and probabilists. The reason for this wide interest is that (exotic) options can have any terms the traders agree to. That is to say, the contract's terms are left up to the imagination of the user. There have been some recent attempts to unify the theory of option pricing, Malliavin calculus for example, but as yet a general analytical approach has not been agreed on; often one resorts to numerical methods.

In this section we shall review some of the more standard (less bizarre) contracts.

1.2.1   American options

 American options are similar to European options. The difference being that American options may be exercised at any time up to expiry, while the European variety may only be exercised at maturity. Since American options give the buyer more freedom than the European options, an American option must be worth at least as much as the corresponding European.

It is beneficial for the buyer of an American option to exercise at any time the stock price is such that a profit (above the cost of the option) is realised. The buyer must select, from among these times, the best time to exercise the option. If the option is exercised too early, the possibility of a greater profit is forfeited. On the other hand, if the option is exercised too late the optimum payout may be missed.

The problem of choosing the best time to exercise is not a trivial one. The same problem arises in other contexts and has become known as *the optimal stopping problem*. It comes under the heading of stochastic control. There has been recent work done in this department on this problem; in particular that done by Alcock and Denman (Alcock & Denman November, 1999) under the supervision of Professor Burrage. In MN480 (s2,1999), the follow-on course to MS479, they developed an algorithm called the BAD-method which computes an approximate numerical solution to the optimal stopping problem.

1.2.2   Asian options

Asian options have a payoff function which depends on the average value of the underlying over the duration of the contract. They are attractive alternatives to European or American options as their values are less affected by sudden temporary price changes in the underlying. As such they are usually used when a portfolio manager wishes to reduce amount of volatility and make hedging of a particular stock easier.

There are two possibilities for the form of dependence of the payoff on the average stock price. The payoff could either depend on the difference between the average price A and the price at maturity S_T, or it could depend on the difference between A and some predetermined strike K. These are distinguished as *floating strike* and *fixed strike* respectively.

There is also freedom in the choice of definition for the average A. The most commonly used is the *arithmetic average*

$$A = \frac{1}{T} \int_0^T S_t dt,$$

which is simply the mean stock price over the interval. At present, there is no exact formula for the value of an Asian option with arithmetic averaging. If however, one chooses the average to fit nicely into the framework of the model, then an exact formula can be reached. For example, there is an exact pricing formula for Asian options which use *geometric averaging*

$$A = \frac{1}{T} \int_0^T \ln(S_t) dt.$$

1.2.3 Multi-reset options

Reset options have an added extra dimension above standard option types. These options start out with a specified strike, but the strike may be changed (reset) throughout the contract's duration. The strike price for a reset option is reset equal to the stock price on the reset date if it is beneficial to the option holder to do so. The reset feature increases the option's value over a standard option significantly. These options may be European in style with fixed reset dates, or they may be American in style giving the holder the freedom to choose their reset dates.

If I hold, for instance, a European reset put option and the stock price is above the strike, K on the reset date, T_1, then I would exercise my option to reset the strike to the current stock price S_{T_1}. In doing so I have converted a put option which was out of the money, to an option which is at the money and is more likely to be valuable at expiry time T. The payoff function for this option is

$$P_T = \begin{cases} S_{T_1} - S_T, & \text{if } S_{T_1} > K \text{ and } S_T \leq S_{T_1}; \\ K - S_T, & \text{if } S_{T_1} \leq K \text{ and } S_T \leq K; \\ 0, & \text{otherwise.} \end{cases}$$

Users of reset options of the American mould have the added quandary of choosing an optimal reset date. For the single reset case this problem reduces to the optimal stopping problem briefly discussed in Section 1.2.1. In essence the problem can be viewed as an optimal control problem where one has control over the timing of the reset, and the objective of resetting to the most favourable price.

Recent research has been done within this department on the case of multiple resets, the so-called multi-reset options. Kerr, Burrage & Gray (Submitted for review) adapted a numerical method based on a binomial lattice to the problem of pricing American-style multi-reset puts. In addition they develop a search algorithm on a finite lattice which seeks the approximate points of optimal resetting.

1.3 Financial jargon

Here is a crossword containing some of the terms which are used in the mathematical finance literature.

1 ↓ a falling market;

1 ↗ a rising market;

1 → traded as currency;

2 ↓ periodical payment to the holders of some shares;

3 ↓ Expiry date, exercise date, the date on which the option must be exercised or becomes void and ceases to exist, T;

4 ↓ to bet;

5 → to reduce the risk of loss; often results in a reduction of the possibility for profit;

6 ← the initial value of the contract, C_0;

7 ↓ an asset; the stock on which the contract's value depends, S_T;

8 → exercise price; the amount for which the underlying can be bought or sold, K;

9 → the current price at which the underlying is being bought and sold, S_t;

10 ← there exists a strategy which has the possibility for making a profit, but no risk of loss;

FINANCIAL CALCULUS
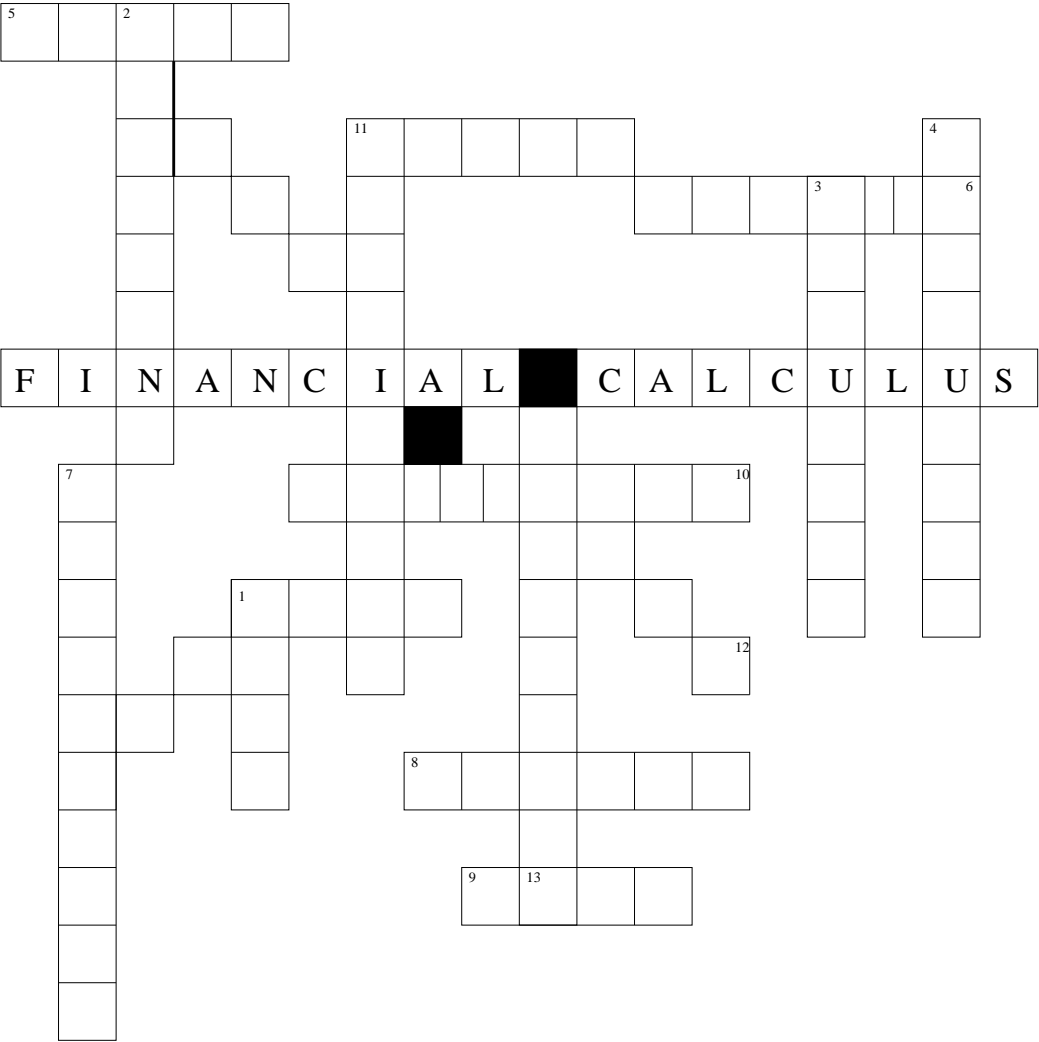
11 → a payment which will be made according to a contract;

11 ↓ depending on the behaviour of the market;

12 ↖ a security whose value depends on existing, underlying, securities; a contingent claim;

13 ↑ a collection of stocks, bonds, and contracts.

1.4 Exercises

1. What does the profit diagram for European put option look like?

2. How can a bear spread be constructed using call options?

3. Suppose I have constructed a portfolio by selling one call option, buying a put option, selling one unit of stock short, and holding K dollars. Here K is the strike price of both options, and both the options have the same expiry date T. By examining the cash flows in each of the cases $S_T > K$ and $S_T < K$ determine the payoff function of this portfolio.

4. A particular option strategy, called a *butterfly spread*, has a payoff described in Figure 1.5. List a portfolio of options which would give this payoff. Under what scenario would this strategy be useful?

5. Fill in Table 1.1. All contracts are on the same stock and mature at the same time.

6. How much would it cost, using the contracts in Table 1.1, to set up the strategy specified in Question 4?
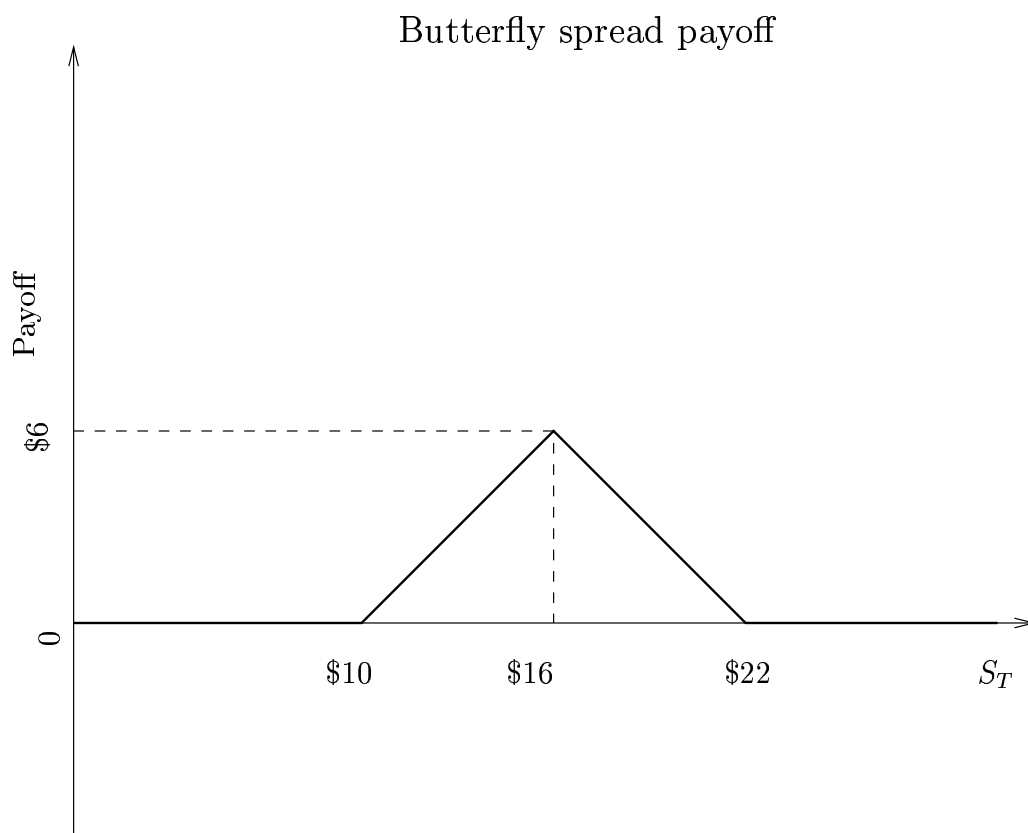
Butterfly spread payoff

Figure 1.5: Payoff diagram for a butterfly spread

contract type	strike price	current value
call option	10	7.15
call option	16	4.50
call option	22	
call option	x	1.00
put option	10	2.15
put option	16	
put option	22	10.00
put option	x	11.00
forward contract	16	

Table 1.1: