**Aim**: find $u(x)=v(r)$, ($x \in \mathbb{R}^n$, $n \geq 2$)

$r=|x|$ s.t. $\Delta u = 0$:

i.e., find a radially symmetric harmonic $f^{(n)}$.

$$\boxed{v''(r)+\frac{n-1}{r}v'(r)=0 \quad (++)}$$

$r>0$.

Put $w=v'$.

$(++) \Rightarrow \quad w'+\frac{n-1}{r}w=0.$

$\Rightarrow \quad \frac{w'}{w}=\frac{1-n}{r} \qquad w\neq 0$

i.e. $(\log w)'=\frac{1-n}{r} \qquad \log=\ln$

$\Rightarrow \quad \log w=(1-n)\log r+C_1$

$\Rightarrow w = a$ …

$v'$

integrate …

$v=\begin{cases} b \ldots \\ \frac{b}{r^{\ldots}} \ldots \end{cases}$

$\rightsquigarrow$ fundam…

$\neq 0$

$= \ln$

$\Rightarrow w = a r^{1-n}$, $a = e^{c_1}$

integrate $\Rightarrow$

$$V = \begin{cases} b \log r + c & n = 2 \\ \dfrac{b}{r^{n-2}} + c & n \geq 3 \end{cases}$$

$\rightsquigarrow$ fundamental sol$^n$ of

Laplace's eq$^n$ on $\mathbb{R}^n \setminus \{0\}$, given by

$$\Phi(x) = \begin{cases} \dfrac{1}{2\pi} \log|x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha_n} |x|^{2-n} & n \geq 3 \end{cases}$$

$\alpha_n$ = volume of unit ball in $\mathbb{R}^n$

$$= \int_{|x|<1} 1 \, dx_1 \cdots dx_n = \underbrace{\int \cdots \int}_{n \text{ times}}$$

e.g. for $n=3$:

$$\Phi(x) = \frac{1}{4\pi}|x|^{-1}$$

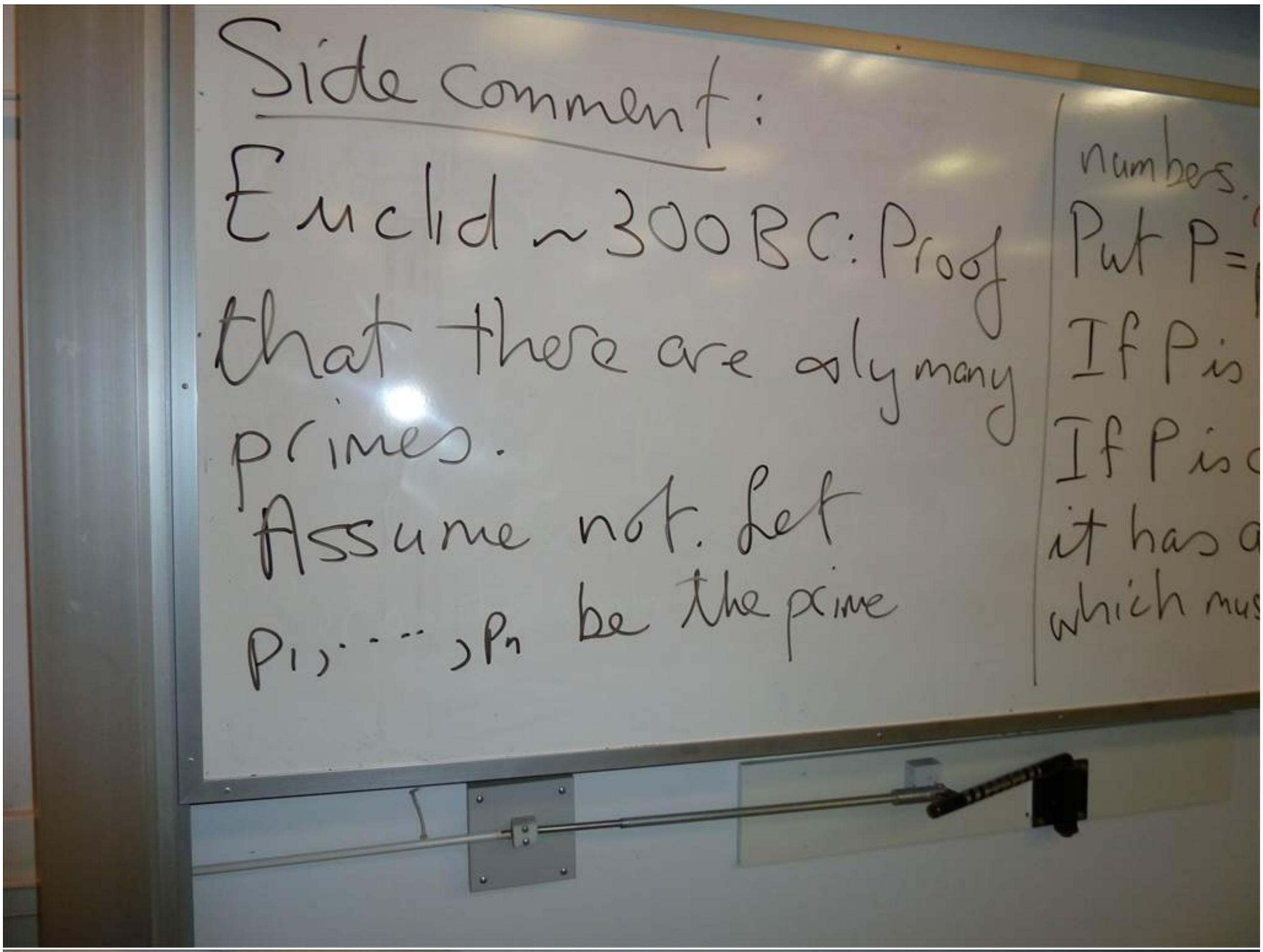
* can show that the convolution

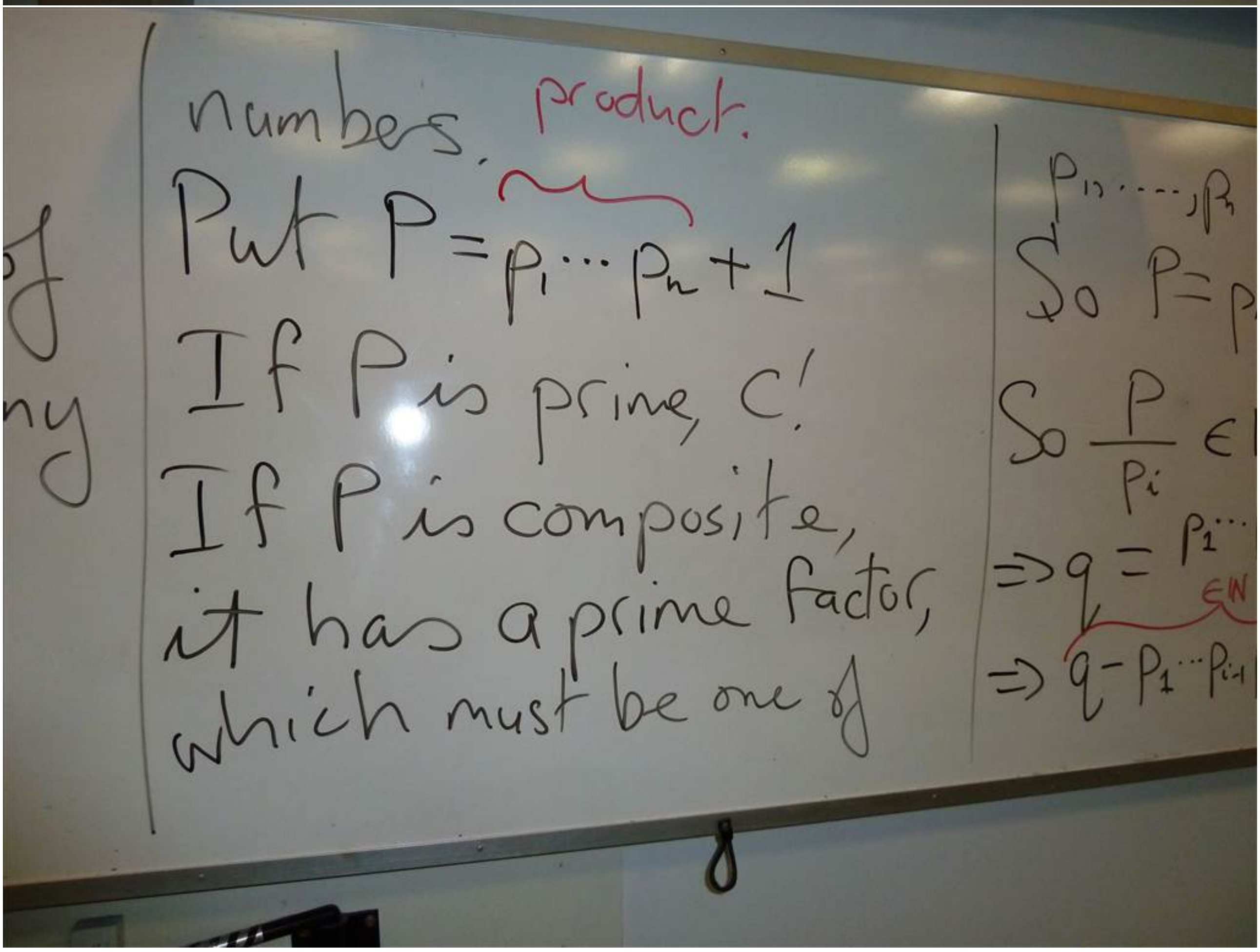
$$u(x) = -\int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \qquad dy_1 \cdots dy_n$$

Solves
(Poisso
nice

Solves $\Delta u = f$
(Poisson's eq$^n$), for $f$
nice enough.


Side comment:
Euclid ~ 300 BC: Proof
that there are ∞ly many
primes.
Assume not. Let
$p_1, \dots, p_n$ be the prime
numbers.


numbers.
product.
Put $P = p_1 \cdots p_n + 1$
If P is prime, C!
If P is composite,
it has a prime factor,
which must be one of
$p_1, \dots, p_n$

$p_1, \dots, p_n$, say $p_i$.

So $P = p_i q$, $q \in \mathbb{N}$, $q > 1$.

So $\dfrac{P}{p_i} \in \mathbb{N}$.

$$\Rightarrow q = p_1 \cdots p_{i-1} p_{i+1} \cdots p_n + \frac{1}{p_i}$$

$$\Rightarrow \underbrace{q - p_1 \cdots p_{i-1} p_{i+1} \cdots p_n}_{\in \mathbb{N}} = \frac{1}{p_i} \notin \mathbb{N}, \text{ all } p_i \geq 2$$

C! $\square$

Examples of PDE (+initial/boundary conditions)

The Dirichlet Problem for Laplace's eq$^n$:

$$(D) \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \ (*) \end{cases}$$

(*) $\Rightarrow$ for $x \in \partial\Omega$, $u(x) = g(x)$.


$\leftarrow$ $u = g$ here.

"extend $g$ to $\Omega$ harmonically."

Sol$^n$s of $\Delta u = 0$ are called harmonic functions.

Comes up in: electrostatics, stedy-state fluid flow, steady-state heat / temperature distrubution, $\mathbb{C}$ analysis.

Classify: 2nd order, linear, homogeneous.

(D) is well posed (for $g$ & $\Omega$ nice enough).

The Cauchy problem for the heat equation on $\Omega \times (0,\infty)$ is:

$$(H)\begin{cases} u_t = \Delta u & \text{on } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = u_0(x) & x \in \partial\Omega. \end{cases}$$

parabolic cylinder