If $u \in C^2(\Omega)$ then $u_{xy} = u_{yx}$

i.e. $D_{12}u = D_{21}u$.

If $u \in C^4(\Omega)$, then

$$D_{1337}u = D_{3137}u = D_{7331}u \text{ etc.}$$

If $\Omega \subset \mathbb{R}^8$, these derivatives are written as

$$D^{(1,0,2,0,0,0,1,0)}u.$$

## Classifying PDE.

$$F(x, u, Du, D^2u, \dots, D^k u) = 0 \quad \odot$$

The order is the order of the highest partial occurring in $\odot$, this is $k$.

(A) The PDE $\odot$ is called linear if it has the form

$\sum_{|\alpha| \leq k} a_\alpha(x)$ …

(i.e. the o…

$L = \sum_{|\alpha| \leq k}$ …

If $f \equiv 0$, …

homogeneous, …

inhomogeneou…

e.g. $x^2 u + \Delta$ …

$$\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x)$$

(i.e. the operator $L = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$ is linear).

If $f \equiv 0$, the eq.$^n$ is **homogeneous**, otherwise **inhomogeneous**.

e.g. $x^2 u + \Delta u = 0$.

$$x^2 D^{(0)} u + 1 \cdot D^{(2,0)} u + 1\, D^{(0,2)} u = 0$$

2nd order linear.

$u_{xxx} + x^7 \sin x \, u = 0$ is 3rd order linear.

(B) (o) is called **semilinear** if it has the form:

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + b(x, u, Du, \cdots D^{k-1} u) = 0$$

e.g.: $x^2 u_{xxx} + (u_{xx})^2 u^7 + \sin^2 x = 0$

3rd order semilinear.

(C) (0) is called quasi-linear if it has the form

$$\sum_{|\alpha|=k} a_\alpha(x,u,Du,\cdots,D^{k-1}u)D^\alpha u + b(x,u,Du,\cdots,D^{k-1}u) = 0.$$

$$(x^2+y^2)(u_{xy}^2+u_{yy}^7)u_{xxx} + u^7 u_{xxy} + \sin(u_{xy}+u_{yy}) + x^7 = 0$$

is 3rd order quasi-linear.

(D) (0) is called fully nonlinear: it depends on the highest order derivatives in a nonlinear fashion

$$u_{xx}^2 + u_{yy}^2 - 2(u_{xy})^2 + 7uu^2 = 0$$

2nd order fully nonlinear.

<u>Warm-up</u> $\Omega = \mathbb{R}^n$: put

$u(x) = v(r)$, where $r = |x|$

= distance from zero to $x$:

can also write as $\|x\|$).

<u>Note</u> $u : \mathbb{R}^n \to \mathbb{R}$;

$v : [0,\infty) \to \mathbb{R}$.

$$r = |x| = (x_1^2 + \dots + x_n^2)^{1/2}$$

<u>AIM</u>: Find $v$ s.t. $\Delta v = 0$,

i.e. $\sum_{i=1}^{n} D_{ii} u = 0$, i.e.,

find a radially symmetric harmonic function.

<u>Note</u>: $D_i r = \dfrac{\partial r}{\partial x_i}$

$= \frac{1}{2}(x_1^2 + \dots + x_n^2)^{-1/2} \cdot 2x_i$

$= x_i / r \qquad (r \neq 0)$

Chain rul

$D_i u = \dfrac{\partial u}{\partial x_i}$

$D_{ii} u = D$

$= D$

$= D$

Using $D_i(f$

i.e. pro

Chainrule $\Rightarrow$

$$D_i u = \frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_i}$$

$$= v'(r)\, x_i/r \qquad \oplus$$

$$D_{ii} u = D_i(D_i u)$$

$$= D_i\left(v'(r)\frac{x_i}{r}\right) \text{ via } \oplus$$

$$= D_i(v'(r))\cdot\frac{x_i}{r} + v'(r)\cdot D_i\left(\frac{x_i}{r}\right)$$

using $D_i(fg) = (D_i f)g + f\, D_i g$.

i.e. product rule.

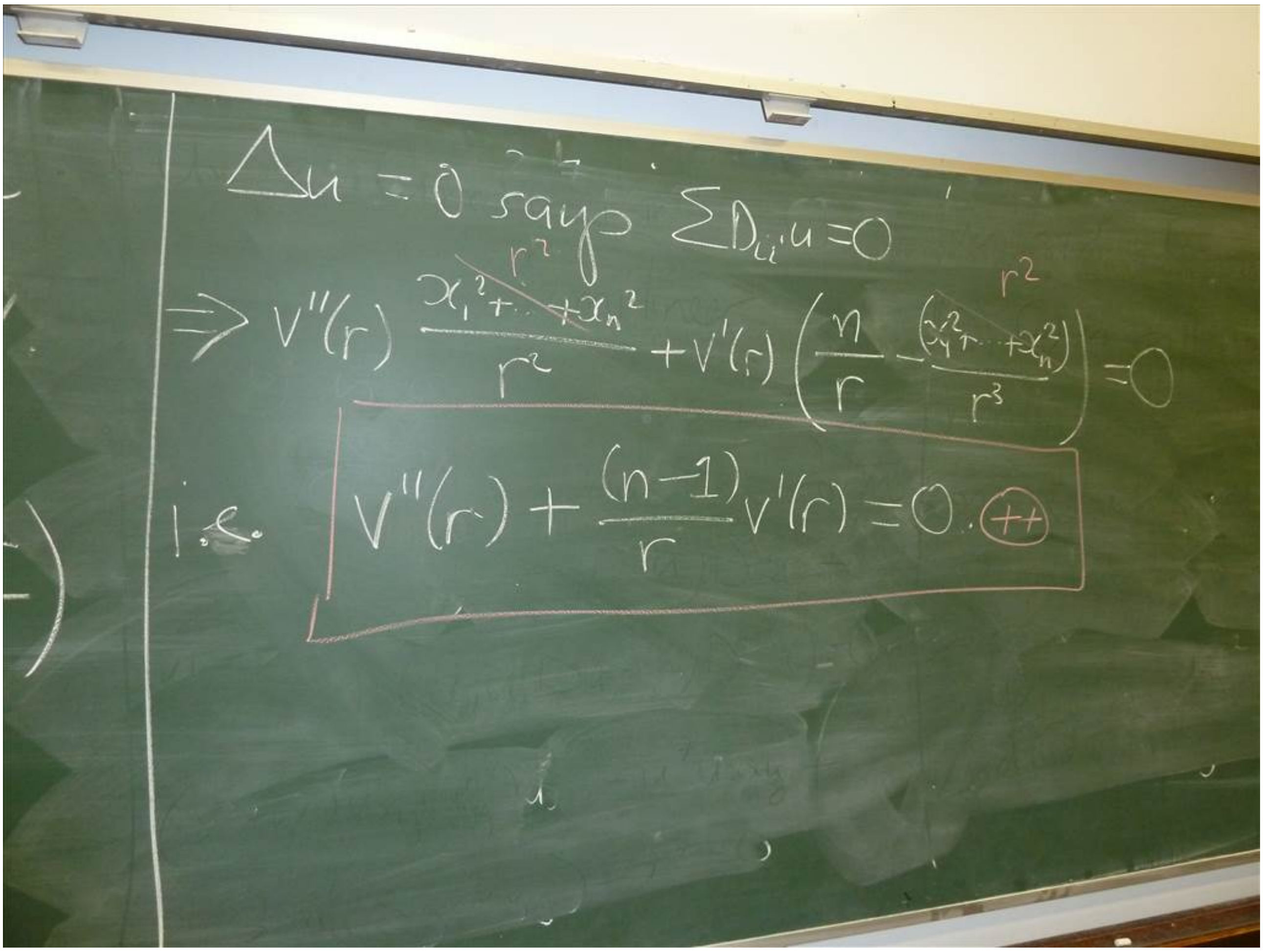
To evaluate $D_i(v'(r))$, use

$\oplus$, but with $v'$ in place of $v$.

$$\Rightarrow D_{ii}u = \left((v')'(r)\frac{x_i}{r}\right)\frac{x_i}{r}$$

$$+ v'(r)\cdot\left(\frac{r\cdot 1 - x_i\cdot x_i/r}{r^2}\right)$$

$$= v''(r)\frac{x_i^2}{r^2} + v'(r)\cdot\left(\frac{1}{r} - x_i^2/r^3\right)$$


$\Delta u = 0$ says $\sum D_{ii} u = 0$
$\Rightarrow v''(r) \frac{x_1^2 + \dots + x_n^2}{r^2} + v'(r)\left(\frac{n}{r} - \frac{(x_1^2 + \dots + x_n^2)}{r^3}\right) = 0$
$r^2$
$r^2$
i.e. $v''(r) + \frac{(n-1)}{r} v'(r) = 0$. (++)