Fix $n \geq 2$, let Ω be a domain in $\mathbb{R}^n$, & $u : \Omega \to \mathbb{R}$

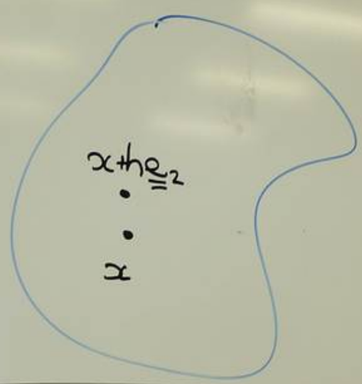
Recall: for $x \in \Omega$

$$\frac{\partial u}{\partial x_i}(x) = \partial_i u(x) = u_i(x) = \lim_{h \to 0} \frac{u(x + h\underline{\underline{e}}_i) - u(x)}{h}$$

IF THE LIMIT EXISTS!

$u(x_1, x_2, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n)$

here $x = (x_1, \dots, x_n)$



here $\underline{\underline{e}}_i$ is the unit vector in the direction of +ve x_i.

In $\mathbb{R}^3$: $\underline{\underline{e}}_1 =$

$\frac{\partial u}{\partial x_i}$ is also

or u_{x_i}.

Similarly:

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \frac{\partial}{\partial x_i}$$

$= (u_{x_j}$

$= u_{x_j}$

IF THE LIMIT EX

In $\mathbb{R}^3$: $\underline{\underline{e}}_1 = \underline{\underline{i}}$, $\underline{\underline{e}}_2 = \underline{\underline{j}}$, $\underline{\underline{e}}_3 = \underline{\underline{k}}$.

$\frac{\partial u}{\partial x_i}$ is also written as $D_i u$ or u_{x_i}.

Similarly:

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_j}(x)$$

$$= (u_{x_j})_{x_i}(x)$$

$$= u_{x_j x_i}(x)$$

NOTE ORDER!

IF THE LIMIT EXISTS.

MULTIINDEX NOTATION

$\mathbb{N}$ = natural numbers = $\{1, 2, 3, \dots\}$

$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$

A <u>multiindex</u> is a vector of the form $(\alpha_1, \dots, \alpha_n)$, where each $\alpha_i \in \mathbb{N}_0$. The <u>length</u> of α is n

(i.e., the number of components) & the order of α, denoted by $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Given such a multiindex α, & $u : \Omega \to \mathbb{R}$, Ω a domain in $\mathbb{R}^n$, set:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}} = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_n}^{\alpha_n} u$$

E.g. on $\mathbb{R}^2$,

$$u_{xx} = D_{x_1 x_1} u = D_{11} u = D^{\overset{\alpha}{\overset{\shortparallel}{(2,0)}}} u.$$

Finally, for $k \in \mathbb{N}_0$, set

$$D^k u = \{ D^\alpha u : |\alpha| = k \}.$$

Convention $D^0 u = u$.

So for $n=2$, $k=1$

$$D^1 u = \{ D^{(1,0)} u, D^{(0,1)} u \}$$
$$= \{ u_x, u_y \}$$

for $n=2$, $k=2$:

$$D^2 u = \{ D^{(2,0)} u, D^{(1,1)} u, D^{(0,2)} u \}$$
$$= \{ u_{xx}, u_{yx}, u_{yy} \}$$

$u_{yx} = u_{xy}$ if u is "nice enough".

We can guarantee $u_{xy} = u_{yx}$ if, for example, all 2nd-order partials of u are continuous.
(This allows us to "swap order of partial differentiation).
Hence:
A f^n^ $g : \Omega \to \mathbb{R}$ is continuously differentiable if g & all partials

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answer

of order 1 exist & are continuous.

We write $f \in C^k(\Omega)$ if all partials of order $\le k$ exist & are cts. continuous

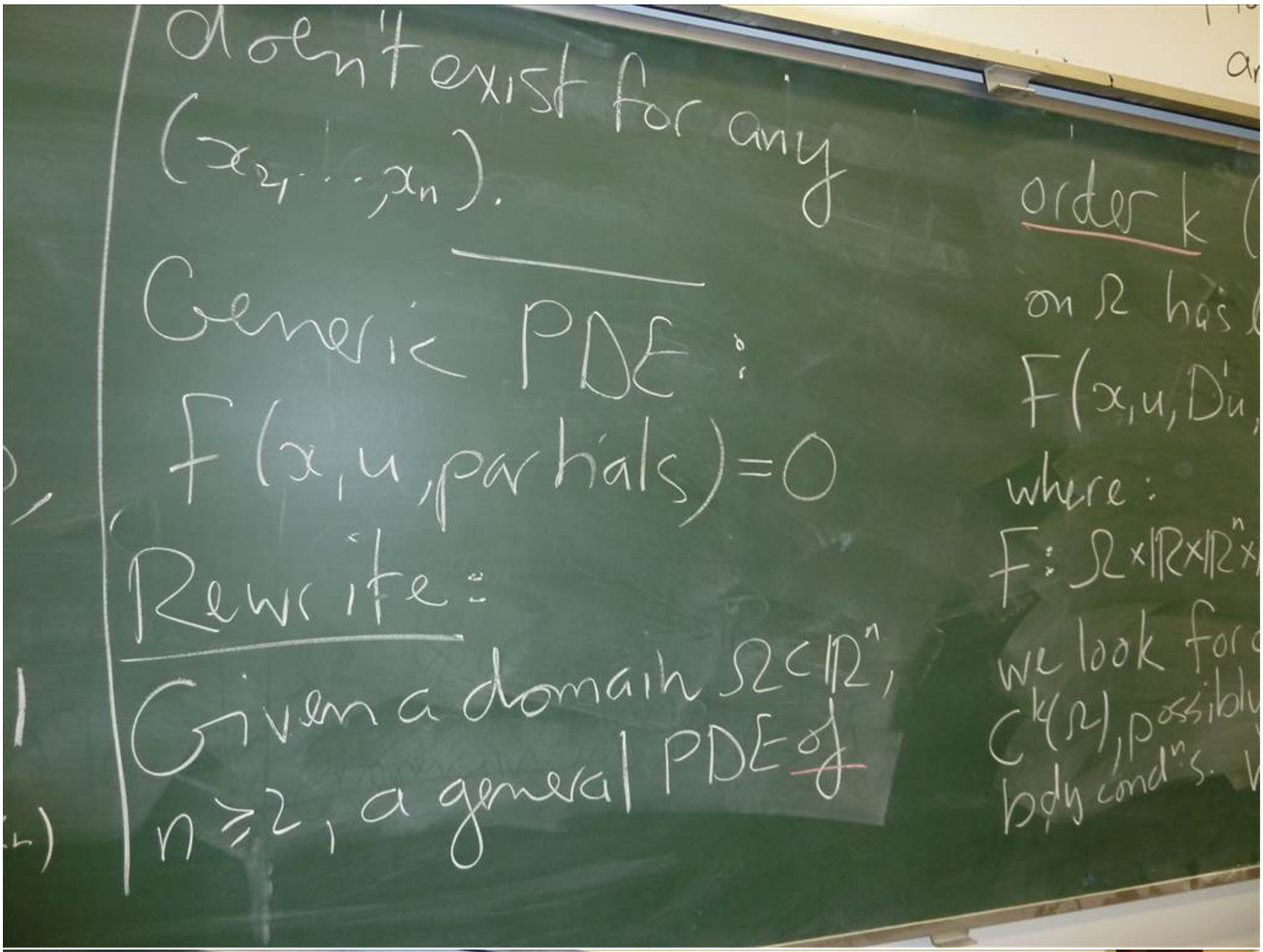
In particular, $f \in C^k \Rightarrow$ "differentiate in any order (up to k times), get the same answer".

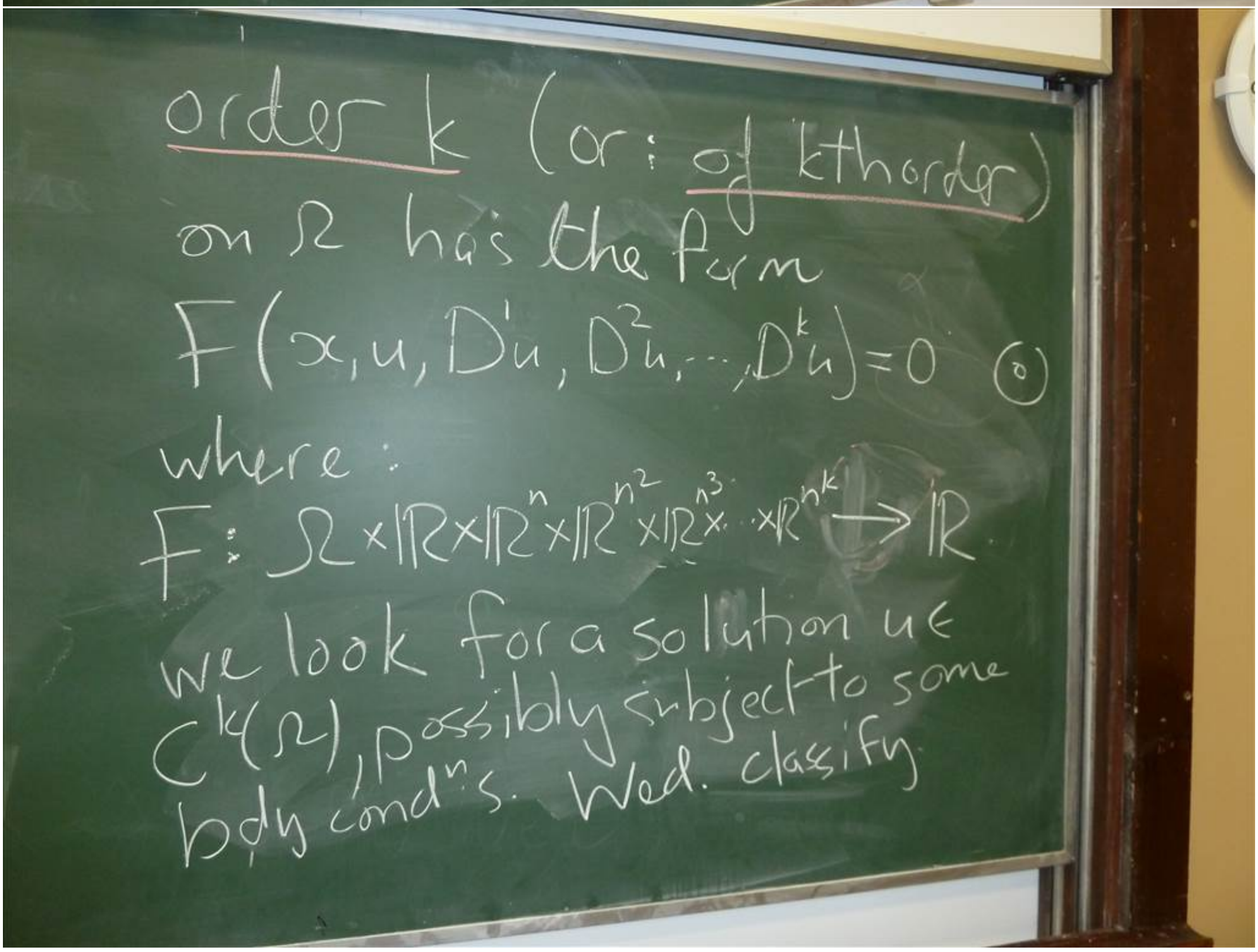
I.e.,

$$D_{i_1 i_2 \cdots i_m} u = D_{\pi(i_1, \dots, i_m)} u$$

if $m \le k$

REMARK: Given any $k \ge 0$, $\exists$ functions that are C^k but not C^{k+1}. E.g. $f(x) = |x_1|$ is cts on $\mathbb{R}^n$, but $\frac{\partial f}{\partial x_1}(0, x_2, \dots, x_n)$


doesn't exist for any
$(x_2, \dots, x_n)$.
Generic PDE:
$F(x, u, \text{partials}) = 0$
Rewrite:
Given a domain $\Omega \subset \mathbb{R}^n$,
$n \geq 2$, a general PDE of


order k (or: of kth order)
on Ω has the form
$F(x, u, D^1u, D^2u, \dots, D^ku) = 0$ ⊙
where:
$F: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}$.
we look for a solution $u \in$
$C^k(\Omega)$, possibly subject to some
bdy cond^{n}s. Wed. classify