Coddington & Levinson

the solution of the initial-value problem

$$x' = f(t,x,\mu) \qquad x(\tau) = \xi$$

was continuous in (t,μ). Actually, the requirement of a Lipschitz condition is too strong; its consequence, the uniqueness of the solution, is sufficient for this important result. The x space is n-dimensional and the μ space is k-dimensional, as in Theorem 7.4.

Theorem 4.1. *Let D be a domain of (t,x) space, I_μ the domain $|\mu - \mu_0| < c$, $c > 0$, and D_μ the set of all (t,x,μ) satisfying $(t,x) \in D$, $\mu \in I_\mu$. Suppose f is a continuous function on D_μ bounded by a constant M there. For $\mu = \mu_0$ let*

$$x' = f(t,x,\mu) \qquad x(\tau) = \xi \tag{4.1}$$

have a unique solution φ_0 on the interval $[a,b]$, where $\tau \in [a,b]$. Then there exists a $\delta > 0$ such that, for any fixed μ satisfying $|\mu - \mu_0| < \delta$, every solution φ_μ of (4.1) exists over $[a,b]$ and as $\mu \to \mu_0$

$$\varphi_\mu \to \varphi_0$$

uniformly over $[a,b]$.

NOTE: Though (4.1) need not have a unique solution for $\mu \neq \mu_0$, nevertheless its solutions are continuous in μ at μ_0.

Proof of Theorem 4.1. The proof will be carried out for the case $\tau \in (a,b)$. The result will first be proved over $|t - \tau| \leqq \alpha$ for some $\alpha > 0$. Choose α small enough so that the region R: $|t - \tau| \leqq \alpha$, $|x - \xi| \leqq M\alpha$ is in D. All solutions of (4.1) with $\mu \in I_\mu$ exist over $[\tau - \alpha, \tau + \alpha]$ and remain in R. Let φ_μ denote a solution. Then the set of functions $\{\varphi_\mu\}$, $\mu \in I_\mu$, is a uniformly bounded and equicontinuous set in $|t - \tau| \leqq \alpha$. This follows from the integral equation

$$\varphi_\mu(t) = \xi + \int_\tau^t f(s,\varphi_\mu(s),\mu)\, ds \qquad (|t - \tau| \leqq \alpha) \tag{4.2}$$

and the inequality $|f| \leqq M$.

Suppose $\varphi_\mu(\bar{t})$ does not tend to $\varphi_0(\bar{t})$ for some $\bar{t} \in [\tau - \alpha, \tau + \alpha]$. Then there exists a sequence $\{\mu_k\}$, $k = 1, 2, \ldots$, for which $\mu_k \to \mu_0$, and corresponding solutions φ_{μ_k} such that φ_{μ_k} converges uniformly over $[\tau - \alpha, \tau + \alpha]$ as $k \to \infty$ to a limit function ψ but $\psi(\bar{t}) \neq \varphi_0(\bar{t})$. From the fact that $f \in C$ on D_μ, that $\psi \in C$ on $[\tau - \alpha, \tau + \alpha]$, and that φ_{μ_k} converges uniformly to ψ, (4.2) for the solutions φ_{μ_k} yields

$$\psi(t) = \xi + \int_\tau^t f(s,\psi(s),\mu_0)\, ds \qquad (|t - \tau| \leqq \alpha)$$

Thus ψ is a solution of (4.1) with $\mu = \mu_0$. By the uniqueness hypothesis, it follows that $\psi(t) = \varphi_0(t)$ on $|t - \tau| \leqq \alpha$. Thus $\psi(\bar{t}) = \varphi_0(\bar{t})$. Thus all solutions φ_μ on $|t - \tau| \leqq \alpha$ tend to φ_0 as $\mu \to \mu_0$. Because of the equicontinuity, the convergence is uniform.