1.7 Cross-sections of a Surface

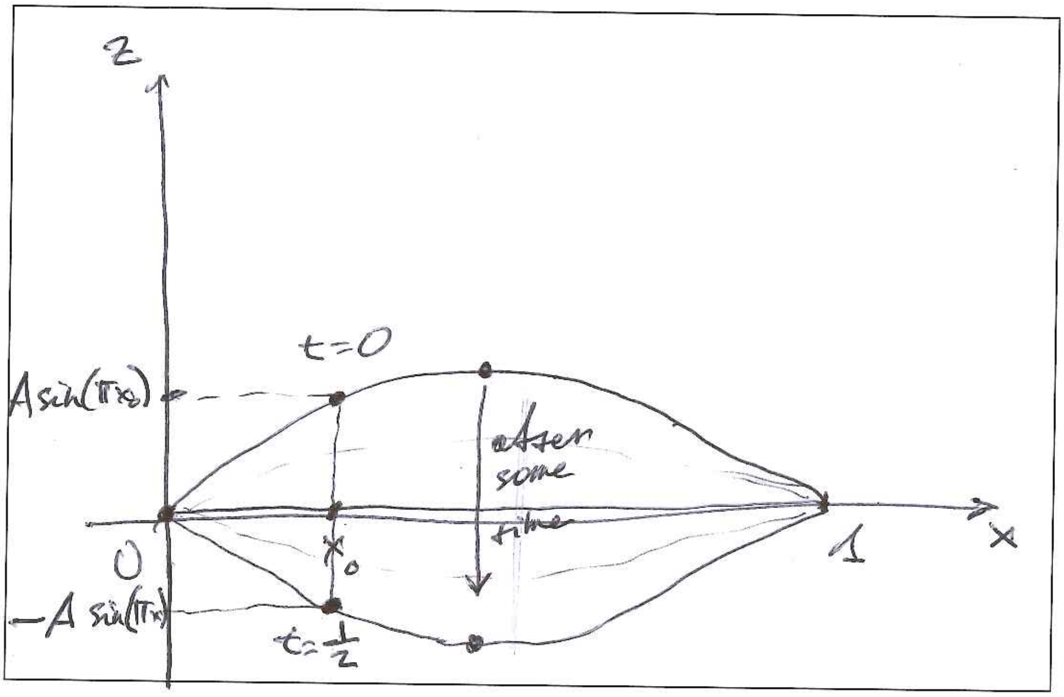
A cross-section is the intersection of a surface with a vertical plane such as $y = C$, see also Stewart Section 12.6 (Section 13.6).

Example:

The height z of a vibrating guitar string can be expressed as a function of horizontal distance x, and time t

$$z = f(x,t) = A\sin(\pi x)\cos(2\pi t) \quad \text{where} \quad 0 < x < 1.$$

The snapshots where t is constant are cross-sections of the 'surface'.



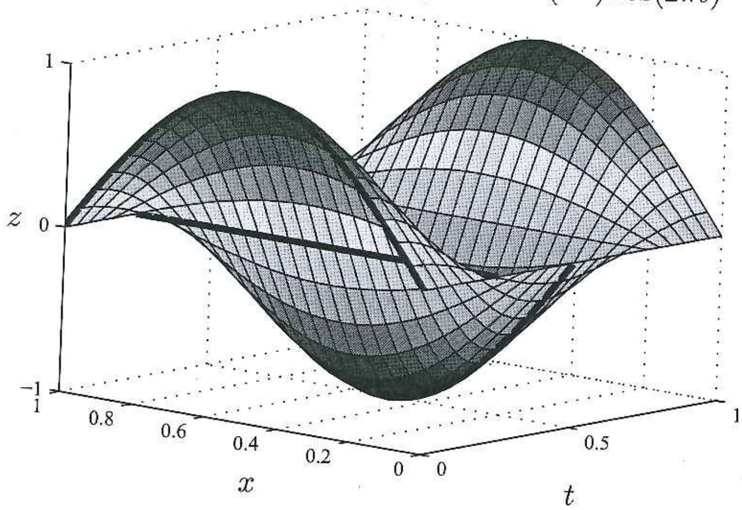
Varying time we get

$$\begin{aligned}
t = 0 &: & z &= A\sin(\pi x)\\
t = \frac{1}{8} &: & z &= \frac{1}{2}\sqrt{2}A\sin(\pi x)\\
t = \frac{1}{4} &: & z &= 0\\
t = \frac{3}{8} &: & z &= -\frac{1}{2}\sqrt{2}A\sin(\pi x)\\
t = \frac{1}{2} &: & z &= -A\sin(\pi x).
\end{aligned}$$

These represent sine curves, with amplitudes between 0 and A.

We can also consider the cross-sections in x. For instance $x = \frac{1}{2}$ (at the top of the sine wave), then $z = A\cos(2\pi t)$ which equals the amplitude of the sine wave.

Vibration of a guitar string: $z = \sin(\pi x)\cos(2\pi t)$

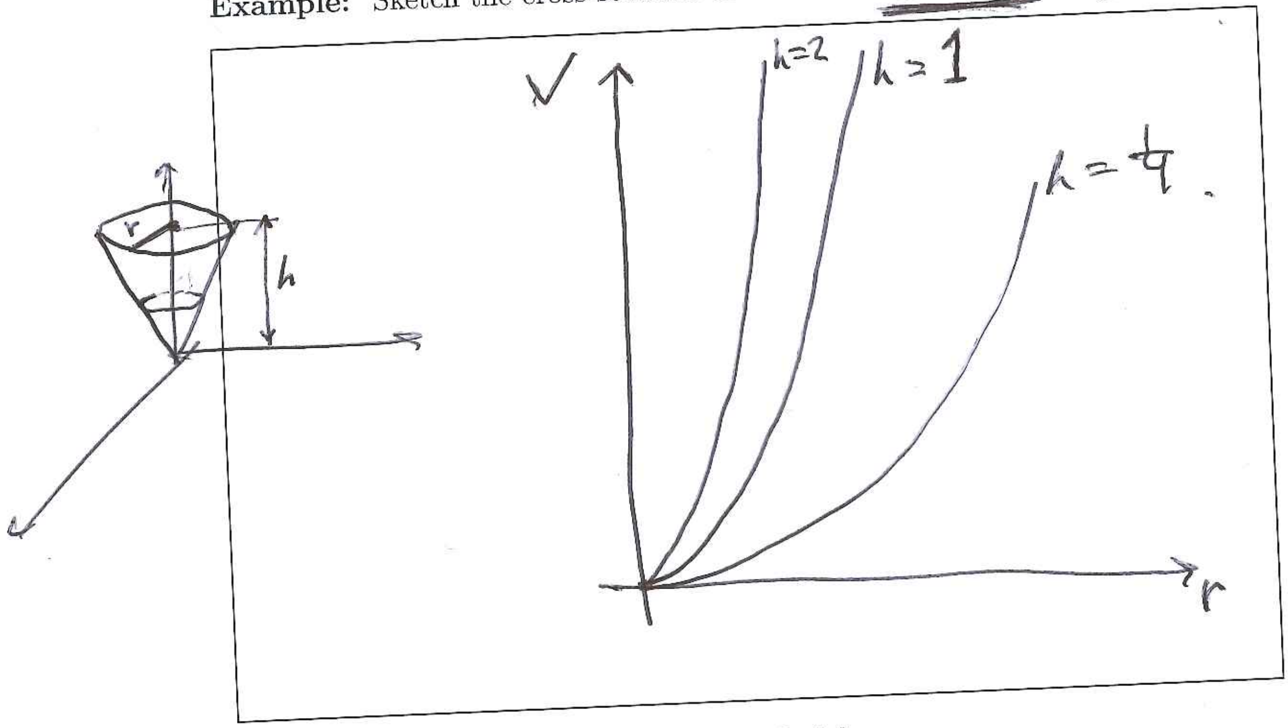


Matlab can be used to make a movie of the 2-dimensional surface by plotting cross-sections at different t values in sequence. The sequence of plots can be stored in a vector and played as a movie using the following code:

```
x=(0:0.25:1);
for j=1:100
   t=j/25;
   z=sin(pi*x)*cos(2*pi*t);
   plot(x,z);axis([0,1,-1,1]);
   M(j)=getframe;
end
```

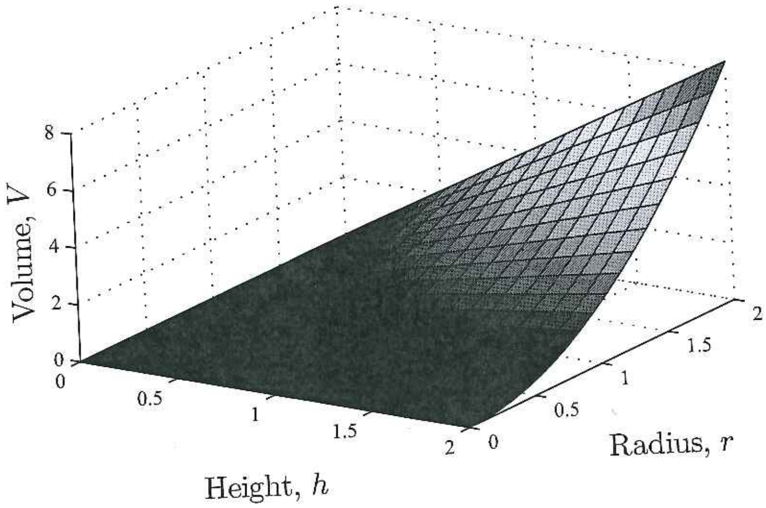
Note: **ezplot** cannot be used to do this because Matlab gets confused about which of t, x is a variable and which is a number.

Example: Sketch the cross-sections of the surface $V = \frac{1}{3}\pi r^2 h$ (volume of a cone).



Here is the 3-dimensional picture from Matlab.

Volume of a cone: $V(r, h) = \frac{1}{3}\pi r^2 h$

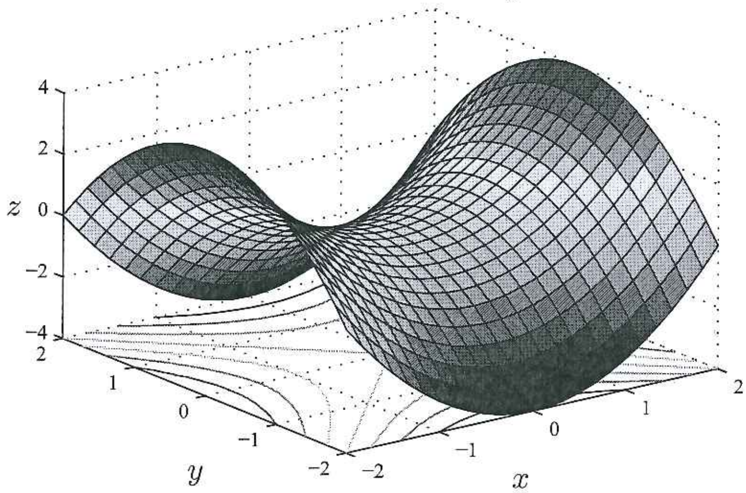


Example: The cross-sections of a saddle $z = x^2 - y^2$ are parabolas. For $y = y_0$ they point up: $z = x^2 - (y_0)^2$; and for $x = x_0$ they point down: $z = -y^2 + x_0^2$.

The surface is tricky to draw, unless you are an equestrian.

Here is the Matlab plot of the saddle $z = x^2 - y^2$ and its contours.

A saddle: $z = x^2 - y^2$



Example: Use cross-sections to sketch the graph of $z = f(x, y) = x^2$.

1.7.1 Main points

- You should be able to construct cross-sections of multivariate functions.
- Cross sections are 2-dimensional graphs.
- Animation of cross-sections is another way to visualise multivariate functions.

2 Partial Derivatives and Tangent Planes

We will need to consider derivatives of functions of more than one variable. To do this, we first check how the familiar concepts of limits and continuity extend to functions of more than one variable. This material is covered in Section 14.2 (Section 15.2) of Stewart.

2.1 Limits and Continuity

2.1.1 Review of the 1-variable case

$(0,1)$

Let $f : D \to \mathbb{R}$ be a function with domain D an open subset of $\mathbb{R}$. For $a \in D$ we say that the limit $\lim_{x\to a} f(x)$ exists if and only if, (i) the limit from the left exists, (ii) the limit from the right exists, and (iii) these two limits coincide, i.e.,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

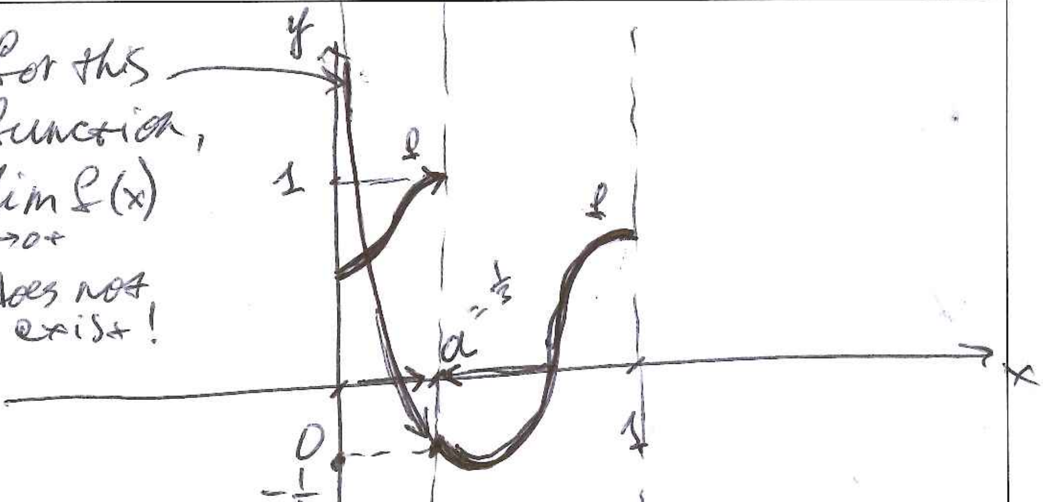
Furthermore, if the limit exists and is equal the actual value of f at a, i.e., if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a),$$

we say that f is continuous at $x = a$.

If f is continuous on all of D we say that f is a continuous function on D.

For this function, $\lim_{x\to 0+} f(x)$ does not exist!



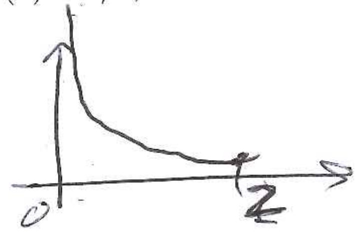
$\lim_{x\to\frac{1}{3}} f(x) = -\frac{1}{4}$

for graph: $\lim_{x\to\frac{1}{3}+} f(x) = -\frac{1}{4}$, NO LIMIT!

$\lim_{x\to\frac{1}{3}-} f(x) = 1$.

We can also consider the limit for points on the boundary of the domain D of a function. For example, if $f : (0,2) \to \mathbb{R}$ is defined by $f(x) = 1/x$, then

$$\lim_{x \to 2^-} f(x) = \frac{1}{2}$$



but

$$\lim_{x \to 0^+} f(x)$$

does not exist.

Another instructive example is $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ given by $f(x) = x^2$. The domain is now the punctured real line, i.e., $D = (-\infty, 0) \cup (0, \infty)$, but

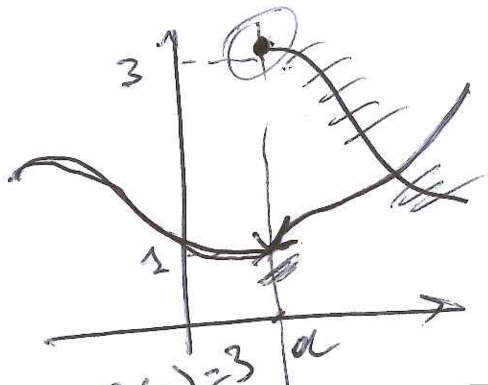
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0.$$

In this situation we also say that $\lim_{x\to 0} f(x)$ exists and in fact one can *fix the hole* by defining $f(0) = 0$, to extend f to a continuous function on all of $\mathbb{R}$.

Important remark: Never, ever compute $\lim_{x\to a} f(x)$ by blindly substituting $x = a$ in f. For example, if

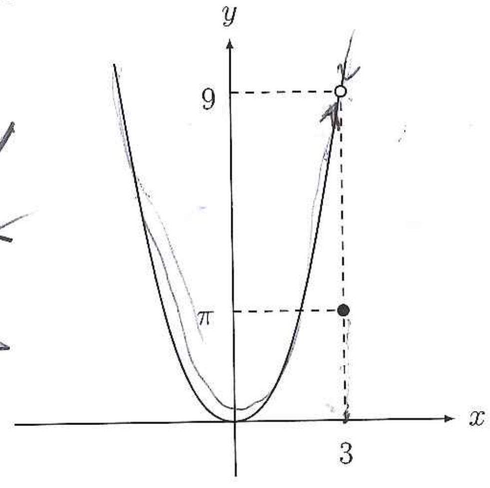
$$f(x) = \begin{cases} x^2 & \text{for } x \neq 3 \\ \pi & \text{for } x = 3, \end{cases}$$

then $\lim_{x\to 3} f(x)$ exists and is given by 9 which is not equal to π: the function f is not continuous at $x = 3$.



$f(a) = 3$

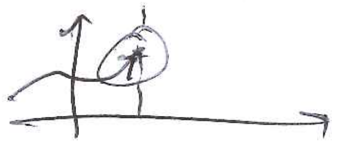
$\lim_{x\to a} f(x) = 1$



As a second example, if $f : D \to \mathbb{R}$ with $D = \mathbb{R}\backslash\{1\}$ is given by

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)} = x+1.$$

then $\lim_{x\to 1} f(x) = 2$. Those (and there will be some) who write "This limit gives $0/0$ which does not exist" should hang their heads in shame.

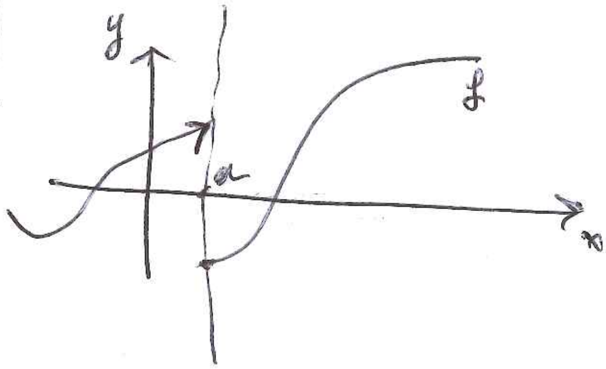


2.1.2 Multivariable limits

When f is a function of more than one variable, the situation is more interesting. There are more than two ways to approach a given point of interest. Consider the function

$$\boxed{f(x,y) = \frac{x^2}{x^2+y^2}}$$

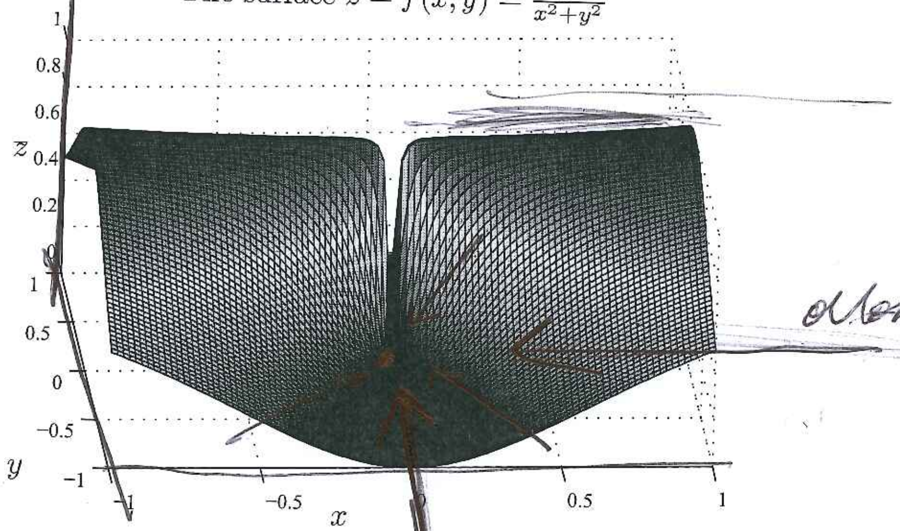
with domain given by $\mathbb{R}^2\backslash\{(0,0)\}$.



To see the graph of f in Matlab, type

```
ezsurf('(x^2/(x^2+y^2)')
```

The surface $z = f(x,y) = \frac{x^2}{x^2+y^2}$



along $y=0$

along $x=0$.

Next we consider the limit as $(x,y) \to (0,0)$.

(i) Approaching the origin along $y = 0$:

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = (\text{along } y=0) = \lim_{x\to 0} \frac{x^2}{x^2+0} = 1.}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = (\text{along } x=0) = \lim_{y\to 0} \frac{0}{0+y^2} = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \text{ does not exist!}$$

(ii) Approaching the origin along $x = 0$:

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

NO

In general, for the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, it is necessary that every path in D approaching (a,b) (the point (a,b) itself may or may not be in D) gives the same limiting value. This gives us the following method for finding if a limit does not exist.

Test for showing no limit exists

If $\begin{cases} f(x,y) \to L_1 & \text{as } (x,y) \to (a,b) \text{ along the path } C_1 \in D \\ f(x,y) \to L_2 & \text{as } (x,y) \to (a,b) \text{ along the path } C_2 \in D \end{cases}$

such that $L_1 \neq L_2$, then the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

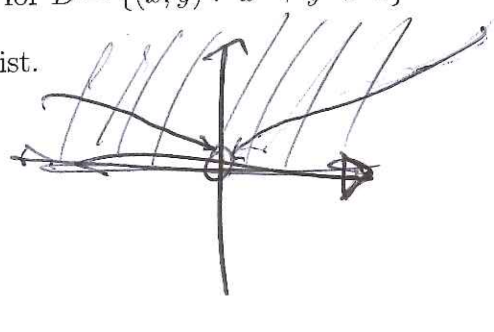
Important remark: The above notation is somewhat deficient and perhaps one should write

$$\lim_{(x,y)\to_D(a,b)} f(x,y)$$

to indicate that only paths in D terminating in (a,b) (which itself may or may not be in D) are considered. For example, if $f(x,y) = x^2 + y^2$ with $D = \{(x,y) : x^2 + y^2 < 1\}$ then $\lim_{(x,y)\to(1,0)} f(x,y)$ exists and is 1. However, if

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{for } D = \{(x,y) : x^2 + y^2 < 1\} \\ 0 & \text{for } D = \{(x,y) : x^2 + y^2 > 1\} \end{cases}$$

then $\lim_{(x,y)\to(1,0)} f(x,y)$ does not exist.



Example: Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$ and $f : D \to \mathbb{R}^2$ be given by $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Along $x=0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{0-y^2}{0+y^2} = -1,$$

Along $y=0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2-0}{x^2+0} = 1.$$

Limit does not exist!

Example: With the same D as above but now $f(x,y) = \dfrac{xy}{x^2+y^2}$, show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Along $x=0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{0}{0+y^2} = 0.$$

Along $y=0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{0}{x^2+0} = 0.$$

Along $x=y$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+\cancel{y}x^2} = \frac{x^2}{2x^2} = \frac{1}{2}.$$

DOES NOT EXIST!

Important remark: There are infinitely many paths terminating in a given point, say (a,b), in $\mathbb{R}^2$, raising the question if one can ever prove that $\lim_{(x,y)\to(a,b)} f(x,y)$ does exist. The good news is that there are methods that can deal with infinitely many paths simultaneously. The bad news is that these methods (typically ϵ-δ proofs) are not part of this course. See Stewart Sec 14.2 (Sec 15.2), Example 4 for a rigorous ϵ-δ proof that

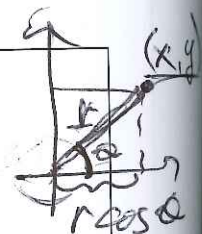
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0,$$

where

$$\boxed{f(x,y) = \frac{3x^2y}{(x^2+y^2)}}$$

and $D = \mathbb{R}^2\backslash\{(0,0)\}$.

Example: Give a ~~non-rigorous~~ proof that the above limit is indeed correct by writing $x = r\cos\theta$ and $y = r\sin\theta$.



Consider $\lim_{(x,y)\to(0,0)} f(x,y)$:

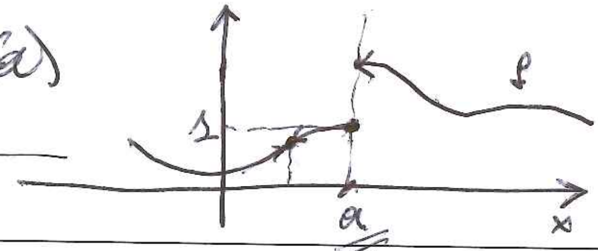
$$f(x,y) = \frac{3\cancel{r^2}\cos^2\theta\, r\sin\theta}{\cancel{r^2}(\cos^2\theta+\sin^2\theta)}$$

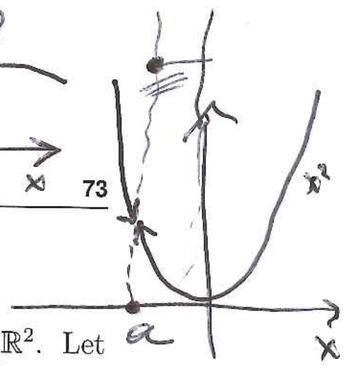
$$= 3\cos^2\theta\, r\sin\theta.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0}(3\cos^2\theta\sin\theta)r$$

$$= 0.$$

$\lim_{x \to a} f(x) = f(a)$





2.1.3 Multivariable continuity

Definition Given a function $f : D \to \mathbb{R}^1$, where D is an open subset of $\mathbb{R}^2$. Let $(a, b) \in D$. Then $f(x, y)$ is continuous at (a, b) if

$$\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b),}$$

i.e., the limit $(x, y) \to (a, b)$ of $f(x, y)$ exists and is equal to $f(a, b)$.

If a function is continuous on all of D we say simply that it is continuous on D. Most of the functions we will consider are continuous. For example, polynomials in x and y are continuous on $\mathbb{R}^2$. As a rule of thumb, if a function with domain D is defined by a single expression it will be continuous on D.

$x^3 + 3x^2 + y^2 + y = f.$

$f = e^{x+y}$

Example: Returning to the first example on page 71, where $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ and $D = \mathbb{R}^2 \setminus \{(0, 0)\}$, is $f(x, y)$ a continuous function?

$f = \begin{cases} x^2 + y^2, & x, y \neq 0, \\ 4, & x, y = 0. \end{cases}$

Problems may arise at $(0,0)$. But $(0,0)$ is excluded from D. So f is continuous in D.

Example: If we edit the above example by instead defining f on all of $\mathbb{R}^2$ by taking $f(0, 0) = 0$, then is f a continuous function? $\Rightarrow$ D

For f to be contin. at D, it has to be cont. at $(0,0)$. Equivalently,
$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$?

2.1.4 Main points

This can't be since lim does not exist!

- You should be able to show when a limit does not exist.
- You should understand continuity of multivariate functions.

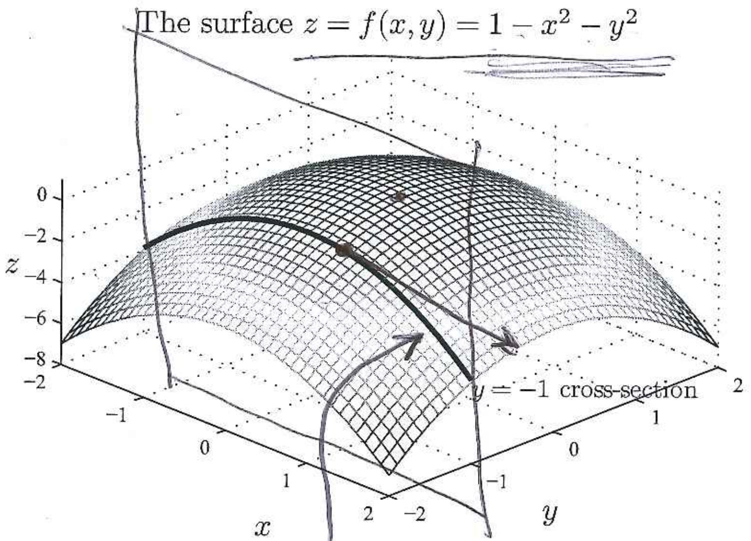
$\frac{d}{dx} f(x)$

2.2 Partial Derivatives

This material is covered in Stewart, Section 14.3 (Section 15.3).

2.2.1 Slope in the x-direction

Consider the surface $z = f(x, y) = 1 - x^2 - y^2$ and the point $P = (1, -1, -1)$ on the surface. Use the "y-is-constant" cross-section through P to find the slope in the x-direction at P.



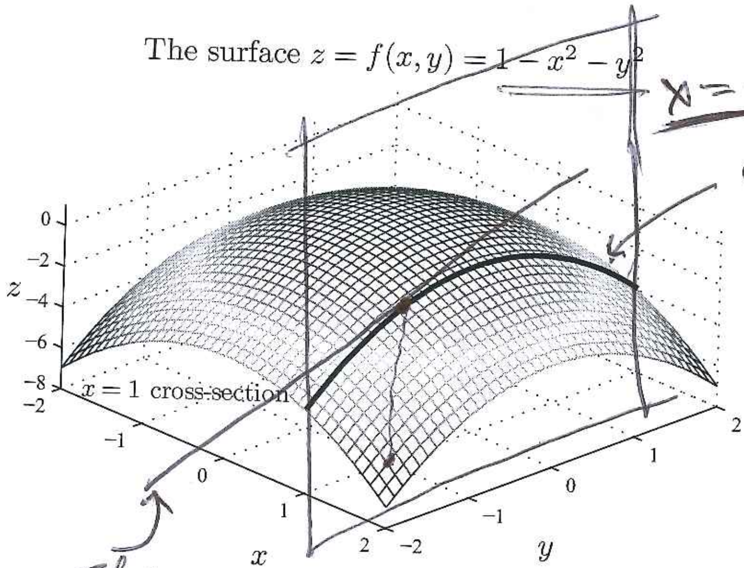
Equation of is $z = 1 - x^2 - (-1)^2$
$= -x^2$,
derivative is $-2x$,
it is slope of the curve.
We denote this derivative by
f_x or $\frac{\partial}{\partial x} f$.

The slope in the x-direction, with y held fixed, is called the partial derivative of f with respect to x at the point (a, b)

$$\frac{\partial f}{\partial x}(a, b) = f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

2.2.2 Slope in the y-direction

Use the "x-is-constant" cross-section to find the slope at $P = (1, -1, -1)$ in the y direction, i.e., where $x = 1$.



$x = 1.$

curve $z = 1-(1)^2-y^2$
$= -y^2,$

Slope $-2y.$

Slope at $P = (1,-1,-1)$ is
$-2 \cdot (-1) = 2.$

Similarly, the slope in the y-direction, with x held fixed, is called the partial derivative of f with respect to y at the point (a, b)

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}.$$

Important remark: Normal rules of differentiation apply, we simply think of the variables being held fixed as constants when doing the differentiation.

Example: Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x,y) = x\sin y + y\cos x$.

$\frac{\partial}{\partial x} f = \sin y + y(-\sin x)$.

$\frac{\partial}{\partial y} f = x\cos y + \cos x$.

Example: Given $f(x,y) = xy^3 + x^2$, find $f_x(1,2)$ and $f_y(1,2)$.

First compute derivatives, then plug in points.

$f_x(x,y) = y^3 + 2x$,

$f_y(x,y) = 3xy^2 + 0$.

$f_x(1,2) = \cancel{4}2^3 + 2\cdot 1 = 10$.

$f_y(1,2) = 3\cdot 1\cdot 2^2 = 12$.

2.2.3 Partial derivatives for $f(x, y, z)$

Example: The volume of a box $V(x, y, z) = xyz$.

If x changes by a small amount, say Δx, denote the corresponding change in V by ΔV. We can easily visualise that $\Delta V = yz\Delta x$.

Change in volume:

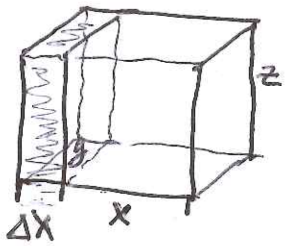
$\Delta V = V_{new} - V$

$= \Delta x \cdot yz,$

$\frac{\Delta V}{\Delta x} = yz = V_x.$

Volume

$V = xyz$



After adding Δx, the volume becomes

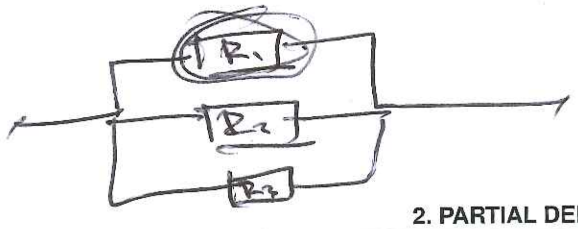
$V_{new} = (x+\Delta x)yz = V + \Delta x yz.$

Therefore,

$$\frac{\Delta V}{\Delta x} = yz.$$

Letting $\Delta x \to 0$ we have $\frac{\partial V}{\partial x} = yz$.

For partial derivatives only one independent variable changes and all other independent variables remain fixed.



Example: Parallel resistance

In an electrical circuit, the combined resistance R, from three resistors R_1, R_2 and R_3 connected in parallel, is

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.}$$

What is the rate of change of the total resistance R with respect to R_1?

Compute $\frac{\partial R}{\partial R_1}$. We have

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{R_1 R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2}$$

$$\frac{\partial R}{\partial R_1} = \left(\frac{R_1 R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2}\right)_{R_1}$$

$$= \frac{R_2 R_3 \cdot (R_2 R_3 + R_1 R_3 + R_1 R_2) - R_1 R_2 R_3 (R_3 + R_2)}{(R_2 R_3 + R_1 R_3 + R_1 R_2)^2}$$

$$= \frac{R_2^2 R_3^2}{(\cdot \ \cdot \ - \ - \ -)^2}$$

2.2.4 Higher order derivatives

The second order partial derivatives of f, if they exist, are written as

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

If f_{xy} and f_{yx} are both continuous, then $\boxed{f_{xy} = f_{yx}.}$

Example: Returning to the example on page 76 for which $f(x, y) = x \sin y + y \cos x$, calculate all of the second order partial derivatives of f and show that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

$f_x = \sin y - y \sin x$,

$f_y = x \cos y + \cos x$,

$f_{xx} = 0 - y \cos x = -y \cos x$,

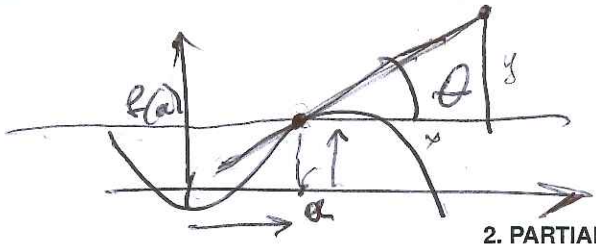
$f_{yy} = -x \sin y$.

$f_{xy} = (f_x)_y = \cos y - \sin x$,

$f_{yx} = (f_y)_x = \cos y - \sin x$.

2.2.5 Main points

- You should know the definition and meaning of partial derivatives.
- You should be able to evaluate partial derivatives of functions.



$y = \tan\theta\, x.$

2.3 The Tangent Plane

This section is covered in Stewart, Section 14.4 (Section 15.4).

2.3.1 Review for $f(x)$

Recall that if $y = f(x)$ then the tangent line at the point $(a, f(a))$ is given by

$$y - f(a) = f'(a)(x - a) \quad \Leftrightarrow \quad \boxed{y = f(a) + f'(a)(x - a)}$$

~~line~~

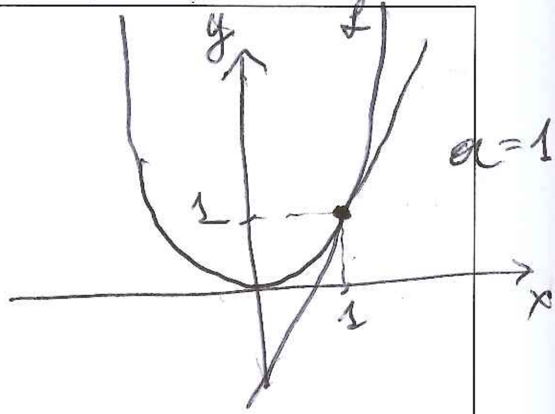
$$\frac{y - f(a)}{x - a} = f'(a).$$

Example: Find the tangent line to $y = f(x) = x^2$ at $x = 1$. $f'(x) = 2x$.

Equation of tangent line:

$y = 1^2 + 2 \cdot 1(x - 1)$

$= 2x - 1.$



2.3.2 Equation for a tangent plane

In general, the equation of the tangent plane to a given surface $z = f(x,y)$ at $(a, b, f(a,b))$, is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b),$$

or, equivalently,

$$\boxed{\overset{\vec{r}}{(x,y,z)} = \overset{\bar{r}_0}{(a,b,f(a,b))} + \lambda\overset{\bar{v}}{(1,0,f_x(a,b))} + \mu\overset{\bar{u}}{(0,1,f_y(a,b))}, \quad \lambda,\mu \in \mathbb{R}.}$$

Indeed, the first two components of this vector equation for the tangent plane imply $\lambda = x - a$ and $\mu = y - b$. Substituting this into the third component gives

$$z = f(a,b) + \lambda f_x(a,b) + \mu f_y(a,b) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

We also note that if we write $\Delta x = x - a$, $\Delta y = y - b$ and $\Delta z = z - f(a,b)$ then the vector equation for the tangent plane may be rewritten as

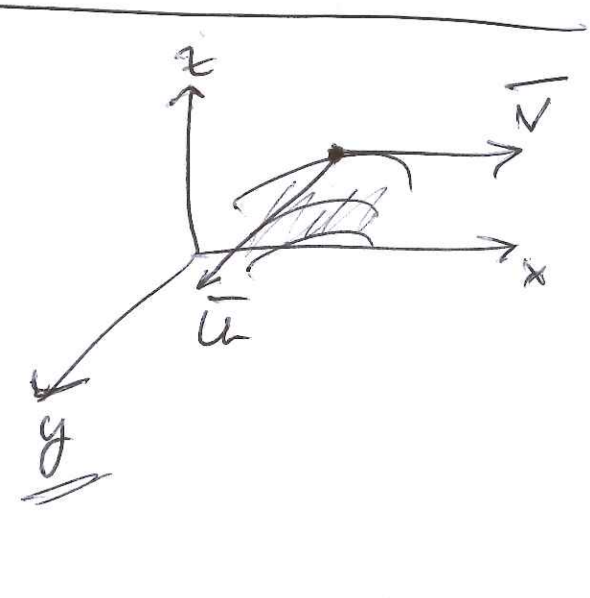
$$(\Delta x, \Delta y, \Delta z) = \lambda(1, 0, f_x(a,b)) + \mu(0, 1, f_y(a,b)), \quad \lambda, \mu \in \mathbb{R}.$$

This shows that if $\Delta x = 1$ and $\Delta y = 0$ (i.e., $\lambda = 1$ and $\mu = 0$) then $\Delta z = f_x(a,b)$ and if $\Delta x = 0$ and $\Delta y = 1$ (i.e., $\lambda = 0$ and $\mu = 1$) then $\Delta z = f_y(a,b)$, matching our interpretation of f_x and f_y as the respective slopes of f in the x or y direction.

$x = a + \lambda,$
$y = b + \mu,$
$z = f(a,b) + \lambda f_x(a,b) + \mu f_y(a,b).$

Choose $\lambda = (x-a)$, $\mu = (y-b)$.

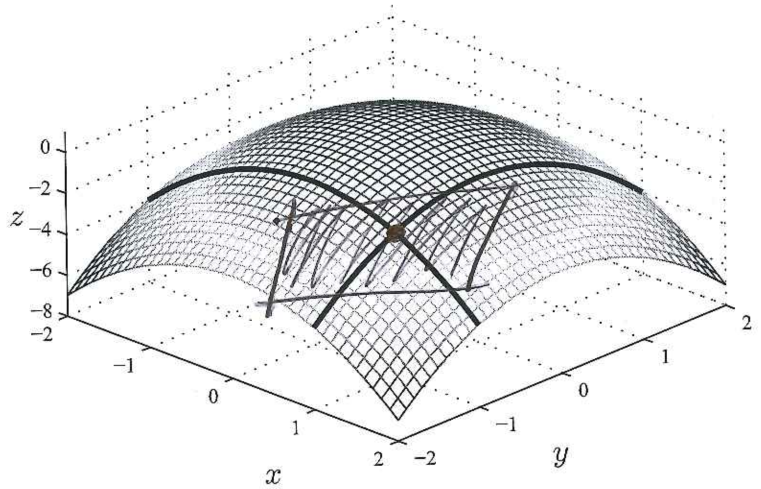
$$\vec{r} = \bar{r}_0 + \lambda\bar{v} + \mu\bar{u},$$



Example: Find the equation for the tangent plane to the surface $\underline{z = 1 - x^2 - y^2}$ $= f(x,y)$ at the point $P = (1, -1, -1)$.

$a \quad b \quad f(a,b) = 1 - 1^2 - (-1)^2 = -1.$

The surface $z = 1 - x^2 - y^2$



Tan. plane:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= -1 + (-2)(x - 1) + 2(y - (-1))$$
$$= -2x + 2y + 3.$$

$$\boxed{\text{Eq.: } z = -2x + 2y + 3.}$$

$f_x(x,y) = -2x, \quad f_y(x,y) = -2y,$

$f_x(a,b) = f_x(1,-1) = -2,$

$f_y(a,b) = 2.$

Example: What is the plane tangent to the surface $z = f(x,y) = 4 - x^2 + 4x - y^2$ at $(1,1)$?

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$,

$a = 1,\ b = 1,\ f(a,b) = f(1,1) = 4 - 1 + 4 - 1 = 6$,

$f_x(x,y) = -2x + 4,\ f_x(1,1) = 2$,

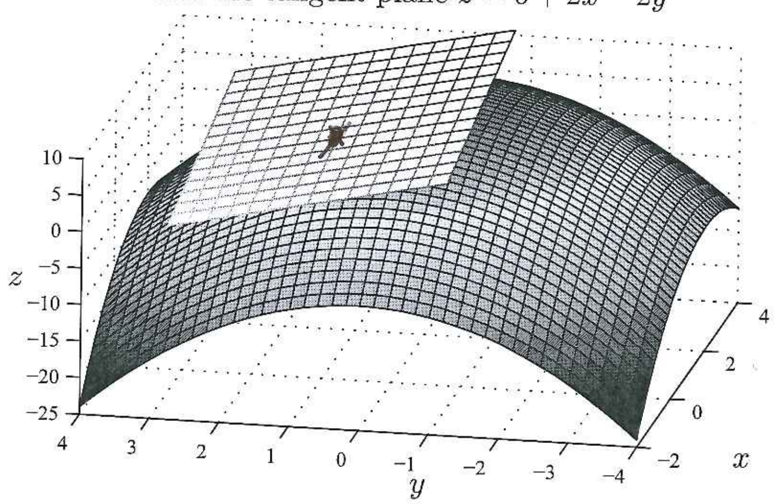
$f_y(x,y) = -2y,\ f_y(1,1) = -2$.

Equation of tan. plane:

$z = 6 + 2(x-1) + (-2)(y-1)$,

$\boxed{z = 2x - 2y + 6.}$

The surface $z = 4 - x^2 + 4x - y^2$
and the tangent plane $z = 6 + 2x - 2y$



Example: Find the tangent plane of $z = f(x, y) = e^{-x^2-y^2}$ at $(x, y) = (1, 3)$.

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$a = 1,\ b = 3,\ f(a,b) = e^{-10}.$

$f_x(x,y) = -2x \cdot e^{-x^2-y^2},$

$f_x(a,b) = -2e^{-10},$

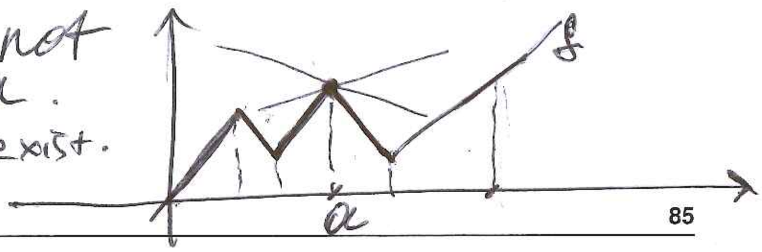
$f_y(x,y) = -2y \cdot e^{-x^2-y^2},$

$f_y(a,b) = -6e^{-10}.$

$z = e^{-10} - 2e^{-10}(x-1) - 6e^{-10}(y-3),$

$\boxed{z = -2e^{-10}x - 6e^{-10}y + 21e^{-10}.}$

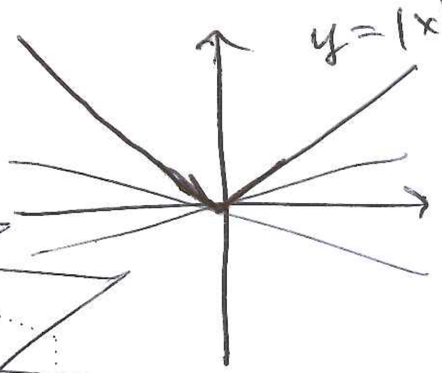
Tangent line not well-defined at a. Derivative does not exist.



2.3.3 Smoothness

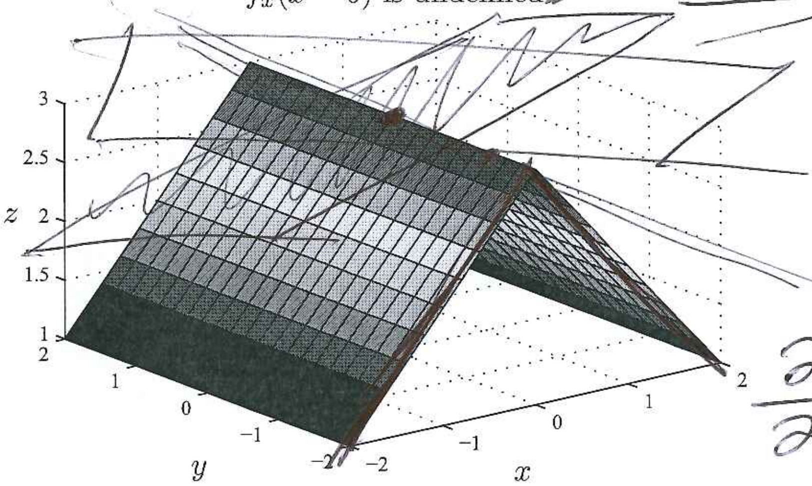
Can we always find partial derivatives and tangent planes?

Example: Simple cusp-like functions are not smooth:



Cusp-like surface: $z = 3 - |x|$

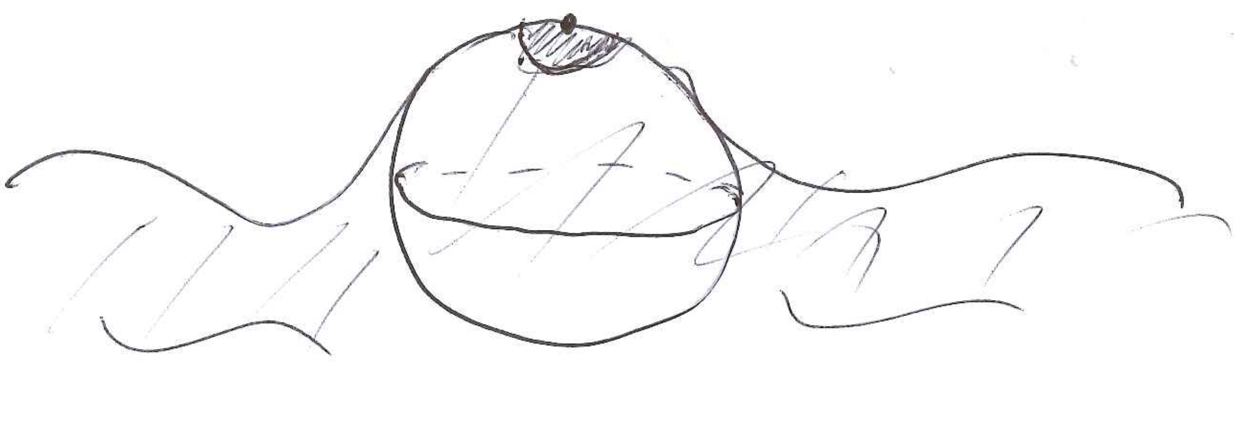
$f_x(x = 0)$ is undefined



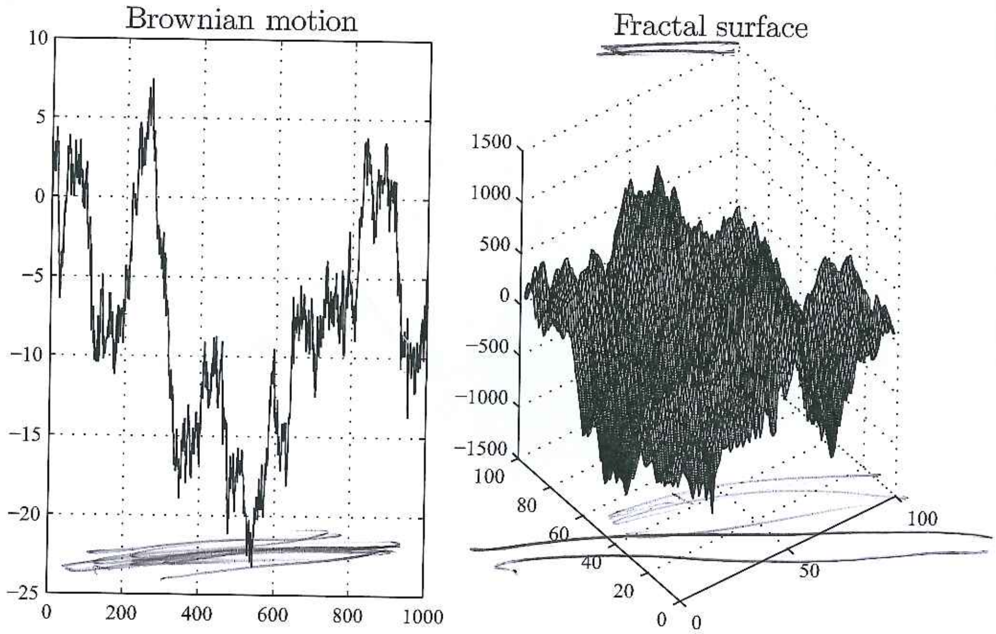
$\frac{\partial f}{\partial x}$ does not exist.

A surface $z = f(x, y)$ is smooth at (a, b) if f, f_x and f_y are all continuous at (a, b). When you zoom in close enough to a smooth surface it looks like a plane.

One way to see this is to look at the contours. The contours of a plane are straight parallel lines, the same perpendicular distance apart. As you zoom into a smooth surface the contours straighten out. This means that close to (a, b) the surface is approximated by a plane; in fact it can be approximated by the tangent plane. Do straight contours imply smoothness?



Example: Brownian motion is not smooth. Look at the figure below. No matter how much you zoom in, it always looks rough—in fact, Brownian motion is a fractal. There are surface-analogues to Brownian motion, demonstrated with the fractal surface below.



2.3.4 Main points

- You should know how to find a tangent plane to a smooth surface, and recognise when a tangent plane or partial derivatives do not exist.