Question 1. Find the extremal for each of the following fix-end point problems:

(i*) \[ \int_{1}^{2} \frac{\dot{x}^2}{t^3} \, dt \quad \text{with} \quad x(1) = 2, \, x(2) = 17. \]

(ii) \[ \int_{0}^{\pi/2} (x^2 - \dot{x}^2 - 2x \sin t) \, dt \quad \text{with} \quad x(0) = 1, \, x(\frac{\pi}{2}) = 2. \]

(iii) \[ \int_{0}^{\pi} (\dot{x}^2 + 2x \sin t) \, dt \quad \text{with} \quad x(0) = x(\pi) = 0. \]

Solution. Case (i):

Let \( f(t, x, \dot{x}) = \frac{\dot{x}^2}{t^3} \). The Euler-Lagrange equation is

\[ \frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) = 0 \]

which implies

\[ - \frac{d}{dt} \left( \frac{2\dot{x}}{t^3} \right) = 0 \quad \Rightarrow \quad \frac{\dot{x}}{t^3} = C \]

where \( C \) is a constant. Then

\[ x = x(t) = kt^4 + l, \quad k = \frac{C}{4} \]

Using \( x(1) = 2 \) and \( x(2) = 17 \), we have

\[ k + l = 2, \quad 16k + l = 17 \]

Solving this linear system yields

\[ k = 1, \quad l = 1. \]

Therefore the extremal is \( x = t^4 + 1 \).

Case (ii):

\[ \int_{0}^{\pi/2} (x^2 - \dot{x}^2 - 2x \sin t) \, dt \quad \text{with} \quad x(0) = 1, \, x(\frac{\pi}{2}) = 2. \]

Let \( f(t, x, \dot{x}) = x^2 - \dot{x}^2 - 2x \sin t \). The Euler-Lagrange equation is

\[ \frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) = 0 \]

which implies

\[ 2x - 2 \sin t + \frac{d}{dt}(2\dot{x}) = 0. \]
This implies

\[(1.1) \quad \ddot{x} + x = \sin t\]

As in the lecture, the fundamental solution of the homogeneous equation \(\ddot{x} + x = 0\) is

\[x_h = A \cos t + B \sin t \quad \text{with constants } A \text{ and } B\]

Next, we are looking for a particular solution to inhomogeneous differential equation (1.1) of the form (modification rule)

\[y(t) = x_p(t) = Lt \cos t + Mt \sin t\]

with two unknown numbers \(L\) and \(M\).

\[\dot{y} = L \cos t - Lt \sin t + M \sin t + M t \cos t,\]

\[\ddot{y} = -L \sin t - L \sin t - Lt \cos t + M \cos t + M \cos t - M t \sin t\]

\[= -2L \sin t + 2M \cos t - Lt \cos t - Mt \sin t\]

Substitute these into DE (1.1) yields

\[-2L \sin t + 2M \cos t - Lt \cos t - Mt \sin t + Lt \cos t + M \sin t = \sin t\]

Which gives \(L = -\frac{1}{2}\) and \(B = 0\). Therefore the general solution of the inhomogeneous equation (1.1) is

\[x(t) = A \cos t + B \sin t - \frac{1}{2} t \cos t\]

Noting \(x(0) = A = 1\) and \(x(1) = B = 2\), the required extremal is

\[x(t) = \cos t + 2 \sin t - \frac{1}{2} t \cos t.\]

Case (iii):

\[\int_0^\pi (\dot{x}^2 + 2x \sin t) \, dt \quad \text{with} \quad x(0) = x(\pi) = 0.\]

Let \(f(t, x, \dot{x}) = \dot{x}^2 + 2x \sin t\). The Euler-Lagrange equation is

\[\frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) = 2 \sin t - \frac{d}{dt}(2\dot{x}) = 0\]

which is equivalent to

\[\ddot{x} = \sin t\]

Then

\[x = x(t) = at + b - \sin t\]

with \(x(0) = 0 = b\) and \(0 = x(\pi) = \pi a\). Therefore \(x(t) = -\sin t\) is the required solution. \(\square\)
Question 2. Find the extremal for each of the following:

(i*) \[ \int_0^2 \frac{\dot{x}^2}{x^3} \, dt \text{ with } x(0) = 1, x(2) = 4. \]

(ii) \[ \int_0^2 \left( \frac{1}{2} \dot{x}^2 + x\dot{x} + x + \dot{x} \right) \, dt \text{ with } x(0) = 0, x(2) = 2. \]

(iii) \[ \int_0^1 \frac{(1 + \dot{x}^2)^{\frac{1}{2}}}{x} \, dt \text{ with } x(0) = 0, x(1) = \sqrt{3}. \]

Solution. Case (i):

\[ \int_0^2 \frac{\dot{x}^2}{x^3} \, dt \text{ with } x(0) = 1, x(2) = 4. \]

Since \( f = \frac{\dot{x}^2}{x^3} \) is independent of \( t \), it follows from the Euler-Lagrange equation that

\[ f - \dot{x} \frac{\partial f}{\partial \dot{x}} = \text{constant}, \]

so

\[ \frac{\dot{x}^2}{x^3} - \frac{2\dot{x}^2}{x^3} = \text{constant} \]

which implies \( \frac{\dot{x}^2}{x^3} = C \) for a constant \( C \). Then

\[ \frac{\dot{x}}{x^{3/2}} = C^{1/2} \Rightarrow -2x^{-1/2} = tC^{1/2} + k \quad \text{for some constant } k. \]

Since \( x(0) = 1, k = -2 \). Since \( x(2) = 4 \), then \( C^{1/2} = \frac{1}{2} \). Then \( x^{-1/2} = 1 - \frac{t}{4} \). Therefore the extremal is

\[ x = \frac{1}{(1 - \frac{t}{4})^2}. \]

Case (ii):

\[ \int_0^2 \left( \frac{1}{2} \dot{x}^2 + x\dot{x} + x + \dot{x} \right) \, dt \text{ with } x(0) = 0, x(2) = 2. \]

Since \( f = \frac{1}{2} \dot{x}^2 + x\dot{x} + x + \dot{x} \) is independent of \( t \), it follows from the Euler-Lagrange equation that

\[ f - \dot{x} \frac{\partial f}{\partial \dot{x}} = \text{constant}, \]

so

\[ \frac{1}{2} \dot{x}^2 + x\dot{x} + x + \dot{x} - \dot{x}(\dot{x} + x + 1) = \text{constant} \]

which implies

\[ -\frac{\dot{x}^2}{2} + x = c, \quad \text{for some constant } c. \]

Hence \( x(t) \geq c \) and

\[ \dot{x}^2 = 2(x - c) \Rightarrow \frac{dx}{dt} = \sqrt{2(x - c)} \]
which is equivalent to
\[ \frac{dx}{\sqrt{2(x-c)}} = dt \]

Integrating yields \( \sqrt{2x-c} = t + k \) for a constant \( k \). Therefore
\[ x - c = \frac{(t + k)^2}{2} \]

Substituting \( x(0) = 0 \) and \( x(2) = 2 \), we have
\[ c + \frac{k^2}{2} = 0, \quad 2 - c = \frac{(2 + k)^2}{2} \quad \Rightarrow k = 0, c = 0. \]

So
\[ x = \frac{t^2}{2}. \]

Case (iii):
\[ \int_0^1 \frac{(1 + \dot{x}^2)^{\frac{1}{2}}}{x} \, dt \quad \text{with} \quad x(0) = 0, \quad x(1) = \sqrt{3}. \]

Since \( f(t, x, \dot{x}) = \frac{(1+\dot{x}^2)^{1/2}}{x} \) is independent of \( t \), it follows from the Euler-Lagrange equation that
\[ f - \dot{x} \frac{\partial f}{\partial \dot{x}} = \text{constant}, \]
so
\[ \frac{(1 + \dot{x}^2)^{\frac{1}{2}}}{x} - \frac{\dot{x}}{x\sqrt{1 + \dot{x}^2}} = c \quad \text{for constant} \ c \]
which implies
\[ \frac{1}{x\sqrt{1 + \dot{x}^2}} = c \]

Then
\[ x\sqrt{1 + \dot{x}^2} = C = \frac{1}{c} \]
for a constant \( C \). We set \( \dot{x} = \tan \theta \). From (4.3), we get \( x = C \cos \theta \) and
\[ \frac{\sin \theta}{\cos \theta} = \tan \theta = \dot{x} = -C \dot{\theta} \sin \theta \quad \Rightarrow 1 = -C \dot{\theta} \cos \theta \]
which is equivalent to
\[ 1 = -C \frac{d}{dt}(\sin \theta). \]

Integrating yields
\[ t + k = -C \sin \theta, \quad \Rightarrow C = -\frac{t + k}{\sin \theta} \]
where \( k \) is a constant. So
\[ x = -\frac{t + k}{\tan \theta} \quad \Rightarrow \quad x = -\frac{t + k}{\dot{x}}. \]
Then
\[ x \dot{x} = -(t + k) \quad \Rightarrow \quad \frac{1}{2} \frac{d}{dt}(x^2) = -(t + k) \]

which implies
\[ \frac{1}{2}x^2 + l = -\frac{(t + k)^2}{2} \]

for a constant \( l \).

Substituting \( x(0) = 0 \) yields \( l = -\frac{k^2}{2} \). Substituting \( x(1) = \sqrt{3} \) yields \( \frac{3}{2} + l = -\frac{(1+k)^2}{2} \). This implies \( k = -2 \) and \( l = -2 \). Therefore
\[ \frac{1}{2}x^2 - 2 = -\frac{(t - 2)^2}{2} \quad \Rightarrow \quad x = \sqrt{4 - (t - 2)^2}. \]