1. Solve the problem of time-optimal control to the origin for each of the following systems

(i) \( \dot{x}_1 = -3x_1 + 2x_2, \quad \dot{x}_2 = 2x_1 - 3x_2 + u \), where \( |u| \leq 1 \)

(ii) \( \dot{x}_1 = x_2, \quad \dot{x}_2 = -3x_1 - 4x_2 + u \), where \( |u| \leq 1 \)

(iii) \( \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u \), where \( |u| \leq 1 \)

Solution Question 1(i).

\[
\begin{align*}
\dot{\tilde{x}} &= A\tilde{x} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ where } |u| \leq 1, \\
A &= \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} \text{ and } \tilde{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{align*}
\]

Cost is \( t_1 \) and has to be in the form

\[
J = \int_0^{t_1} f_0(x_1, x_2, u) dt
\]

to use the PMP. Choose \( f_0 = 1 \). Now \( \psi_0 = -1 \), so

\[
f_0(x_1, x_2, u) = 1, \quad f_1(x_1, x_2, u) = -3x_1 + 2x_2, \quad f_2(x_1, x_2, u) = 2x_1 - 3x_2 + u
\]

(1)

\[
H = \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2
\]

\[
H = -1 + \psi_1(-3x_1 + 2x_2) + \psi_2(2x_1 - 3x_2) + \psi_2 u
\]

\[
\dot{\psi}_1 = -\partial H/\partial x_1 = 3\psi_1 - 2\psi_2, \quad \dot{\psi}_2 = -\partial H/\partial x_2 = -2\psi_1 + 3\psi_2, \text{ that is}
\]

\[
\dot{\psi} = -A^T \psi.
\]

By PMP the optimal control \( u^* \) maximizes \( H \) as a function of \( u \). Since \( H \) is linear in \( u \) and \( |u| \leq 1 \) the maximum value of \( H \) is at \( u^* = 1 \) if \( \psi_2 > 0 \) and at \( u = -1 \) if \( \psi_2 < 0 \); that is,

\[
u^*(t) = \text{sgn } (\psi_2(t)).
\]

Optimal trajectories satisfy

\[
\dot{\tilde{x}} = A\tilde{x} + u^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u^* = \pm 1
\]

Eigenvalues of \( A \) are -1, -5 (STABLE NODE). These are real so \( S = \psi_2(t) \) can change sign at most once \( \Rightarrow \) at most one switch.
# Eqblrm point for \( u^* = 1 \)
\[
\begin{align*}
-3x_1 + 2x_2 &= 0 \\
2x_1 - 3x_2 + 1 &= 0 \\
x_1 &= 2/5 \\
x_2 &= 3/5
\end{align*}
\]
\( P \)

# Eqblrm point for \( u^* = -1 \)
\[
\begin{align*}
-3x_1 + 2x_2 &= 0 \\
2x_1 - 3x_2 - 1 &= 0 \\
x_1 &= -2/5 \\
x_2 &= -3/5
\end{align*}
\]
\( Q \)

# Eigenvectors of \( A \) are:

Case \( \lambda_1 = -1 \)
\[
\begin{pmatrix}
-3 & 2 \\
2 & -3
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix} = - \begin{pmatrix}
u \\
v
\end{pmatrix}, \quad v = u, \quad \begin{pmatrix}1 \\
1
\end{pmatrix} = v_{\sim 1}
\]

Case \( \lambda_2 = -5 \)
\[
\begin{pmatrix}
-3 & 2 \\
2 & -3
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix} = -5 \begin{pmatrix}
u \\
v
\end{pmatrix}, \quad v = -u, \quad \begin{pmatrix}1 \\
-1
\end{pmatrix} = v_{\sim 2}
\]

Diagram of \( u^* = 1 \) paths \( C^+ \). Here \( x(t) = \alpha v_{\sim 1} e^{-t} + \beta v_{\sim 2} e^{-5t} + (2/5)_{\sim 3/5} \)

Diagram of \( u^* = -1 \) paths \( C^- \). Here \( x(t) = \alpha v_{\sim 1} e^{-t} + \beta v_{\sim 2} e^{-5t} - (2/5)_{\sim 3/5} \)

# To get to 0 in minimum time, the phase point \( x(t) \) must travel along a \( C^+ \) path (path with \( u^* = 1 \)) or a \( C^- \) path (path with \( u^* = -1 \)) and can switch from one to another at most once. There is only one \( C^+ \) path going to 0 which we denote by \( P^+O \). Moreover there is only one \( C^- \) path going to 0 which we denote by \( Q^-O \).

# Must arrive at 0
on one of these two paths.
Slope of these curves at 0 is
\[
\frac{dx_2}{dx_1} \sim_{\sim} = \frac{u^*}{0} = \pm \infty
\]

# Put these together:

- Initial state \( w \) on \( P^+O \) or \( Q^-O \), optimal control is \( u^* = 1 \) or \( u^* = -1 \), respectively.
- \( w \) above \( P^+OQ^- \), cannot go to \( 0 \) on a \( C^+ \) path (these go to \( P \)). So go on a \( C^- \) path until \( P^+O \) is reached, then switch
  \[
u^* = \begin{cases} 
-1 & \text{until it reaches } P^+O \\
+1 & \text{afterwards}
\end{cases}
\]
- \( w \) below \( P^+OQ^- \), cannot go to \( 0 \) on a \( C^- \) path (these go to \( Q \)). So go on a \( C^+ \) path until \( Q^-O \) is reached, then switch to \( u^* = -1 \).
  \[
u^* = \begin{cases} 
+1 & \text{until } Q^-O \text{ reached} \\
-1 & \text{afterwards}
\end{cases}
\]

Summary.

\[
u^* = \begin{cases} 
1 & \text{below } P^+OQ^- \text{ & switch on } Q^-O \\
1 & \text{on } P^+O \ldots \text{ no switch} \\
-1 & \text{above } P^+OQ^- \text{ & switch on } P^+O \\
-1 & \& \text{ on } Q^-O \ldots \text{ no switch}.
\end{cases}
\]

Solution Question 1(ii).

\[
\dot{x} = Ax + u \begin{pmatrix} 0 \\
1 \end{pmatrix}, \text{ where } |u| \leq 1,
\]

\[
A = \begin{pmatrix} 0 & 1 \\
-3 & -4 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\
x_2 \end{pmatrix}.
\]

Cost is \( t_1 \) and has to be in the form

\[
J = \int_0^{t_1} f_0(x_1, x_2, u) dt
\]

to use the PMP. Choose \( f_0 = 1 \). Now \( \psi_0 = -1 \), so

\[
f_0(x_1, x_2, u) = 1, \quad f_1(x_1, x_2, u) = x_2, \quad f_2(x_1, x_2, u) = -3x_1 - 4x_2 + u
\]

\[
H = \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2
\]

\[
H = -1 + \psi_1 x_2 + \psi_2 (-3x_1 - 4x_2) + \psi_2 u
\]

\[
\dot{\psi}_1 = -\partial H / \partial x_1 = 3\psi_2, \quad \psi_2 = -\partial H / \partial x_2 = -\psi_1 + 4\psi_2, \text{ that is,}
\]

\[
\dot{\psi} = -A^T \psi
\]
By PMP the optimal control $u^*$ maximizes $H$ as a function of $u$. Since $H$ is linear in $u$ and $|u| \leq 1$ the maximum value of $H$ is at $u^* = 1$ if $\psi_2 > 0$ and at $u = -1$ if $\psi_2 < 0$; that is,

$$u^*(t) = \text{sgn} (\psi_2(t)).$$

Optimal trajectories satisfy

$$\dot{x} = Ax + u^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u^* = \pm 1.$$  

Eigenvalues of $A$ are -1, -3 (STABLE NODE). These are real so $S = \psi_2(t)$ can change sign at most once $\Rightarrow$ at most one switch.
Eqlbrm point for $u^* = 1$
\[
\begin{align*}
    x_2 &= 0 \\
    -3x_1 - 4x_2 + 1 &= 0
\end{align*}
\]
\[
\begin{align*}
    x_1 &= 1/3 \\
    x_2 &= 0
\end{align*}
\] = $P$

Eqlbrm point for $u^* = -1$
\[
\begin{align*}
    x_2 &= 0 \\
    -3x_1 - 4x_2 - 1 &= 0
\end{align*}
\]
\[
\begin{align*}
    x_1 &= -1/3 \\
    x_2 &= 0
\end{align*}
\] = $Q$

Eigenvectors of $A$ are:

Case $\lambda_1 = -1$
\[
\begin{pmatrix}
0 & 1 \\
-3 & -4
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix} = -\begin{pmatrix}
u \\
v
\end{pmatrix},
\begin{pmatrix}
u_1 \\
v_1
\end{pmatrix}
\]
\[
\begin{pmatrix}
u_1 \\
v_1
\end{pmatrix} = \alpha v_1 e^{t} + \beta v_2 e^{-3t} + \begin{pmatrix}1/3 \\
0
\end{pmatrix}
\]

Diagram of $u^* = 1$ paths $\mathcal{C}^+$. Here $x(t) = \alpha v_1 e^{t} + \beta v_2 e^{-3t} + \begin{pmatrix}1/3 \\
0
\end{pmatrix}$

Diagram of $u^* = -1$ paths $\mathcal{C}^-$. Here $x(t) = \alpha v_1 e^{t} + \beta v_2 e^{-3t} - \begin{pmatrix}1/3 \\
0
\end{pmatrix}$

To get to 0 in minimum time, the phase point $x(t)$ must travel along a $\mathcal{C}^+$ path (path with $u^* = 1$) or a $\mathcal{C}^-$ path (path with $u^* = -1$) and can switch from one to another at most once. There is only one $\mathcal{C}^+$ path going to 0 which we denote by $P^+O$. Moreover there is only one $\mathcal{C}^-$ path going to 0 which we denote by $Q^-O$.

Must arrive at 0 on one of these two paths.
Slope of these curves at 0 is
\[ \frac{dx_2}{dx_1} \sim \frac{u^*}{0} = \pm \infty \]

# Put these together:

- Initial state \( w \) on \( P^+O \) or \( Q^-O \), optimal control is \( u^* = 1 \) or \( u^* = -1 \), respectively.
- \( w \) above \( P^+OQ^- \), cannot go to 0 on a \( C^+ \) path (these go to P). So go on a \( C^- \) path until \( P^+O \) is reached, then switch
  \[ u^* = \begin{cases} 
  -1 & \text{until it reaches } P^+O \\
  +1 & \text{afterwards} 
  \end{cases} \]
- \( w \) below \( P^+OQ^- \), cannot go to 0 on a \( C^- \) path (these go to Q). So go on a \( C^+ \) path until \( Q^-O \), then switch to \( u^* = -1 \).
  \[ u^* = \begin{cases} 
  +1 & \text{until } Q^-O \text{ reached} \\
  -1 & \text{afterwards} 
  \end{cases} \]

Summary.

\[ u^* = \begin{cases} 
  1 & \text{below } P^+OQ^- & \text{& switch on } Q^-O \\
  1 & \text{on } P^+O & \text{no switch} \\
  -1 & \text{above } P^+OQ^- & \text{& switch on } P^+O \\
  -1 & \text{& on } Q^-O & \text{no switch.} 
  \end{cases} \]

Solution Question 1(iii).

\[ \dot{x} \sim = \begin{pmatrix} 0 & 1 \\
  0 & -1 \end{pmatrix} x + u \begin{pmatrix} 0 \\
  1 \end{pmatrix}, \text{ where } |u| \leq 1, \]

and

\[ x \sim = \begin{pmatrix} x_1 \\
  x_2 \end{pmatrix}. \]

Cost is \( t_1 \) and has to be in the form

\[ J = \int_0^{t_1} f_0(x_1, x_2, u) dt \]

to use the PMP. Choose \( f_0 = 1 \). Now \( \psi_0 = -1 \), so

\[ f_0(x_1, x_2, u) = 1, \quad f_1(x_1, x_2, u) = x_2, \quad f_2(x_1, x_2, u) = -x_2 + u \quad (3) \]

\[ H = \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2 \]

\[ H = -1 + \psi_1 x_2 + \psi_2 (-x_2) + \psi_2 u \]

\[ \dot{\psi}_1 = -\partial H/\partial x_1 = 0, \quad \psi_2 = -\partial H/\partial x_2 = -\psi_1 + \psi_2. \]
By PMP the optimal control $u^*$ maximizes $H$ as a function of $u$. Since $H$ is linear in $u$ and $|u| \leq 1$ the maximum value of $H$ is at $u^* = 1$ if $\psi_2 > 0$ and at $u = -1$ if $\psi_2 < 0$; that is,

$$u^*(t) = \text{sgn} \left( S(t) \right), \text{ where } S(t) = \psi_2(t).$$

$$\dot{\psi}_1 = 0 \Rightarrow \psi_1 = k, \text{ where } k \text{ a constant}$$

$$\Rightarrow \dot{\psi}_2 = -k + \psi_2 \Rightarrow \psi_2 = le^t + k, \text{ where } l \text{ a constant.}$$

$$\Rightarrow S = \psi_2 = le^t + k \text{ has at most one zero} \Rightarrow \text{at most one switch.}$$

Optimal trajectories satisfy

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u^*, \text{ where } u^* = \pm 1$$

$$\Rightarrow x_2 = Ae^{-t} + u^*, \text{ where } u^* = \pm 1$$

$$\Rightarrow \dot{x}_1 = Ae^{-t} + u^* \Rightarrow x_1 = -Ae^{-t} + u^*t + k, \text{ where } k \text{ a constant.}$$
Diagram of \( u^* = 1 \) paths \( C^+ \). Here \( x(t) = (-Ae^{-t} + t + k, Ae^{-t} + 1) \).

**Notes:**

(i) \( A = 0 \) gives the solution \( x(t) = (t + k, 1) \)

(ii) \( A > 0 \Rightarrow x_2(t) > 1 \) and \( \lim_{t \to \infty} x_2 = 1 \)

(iii) \( A < 0 \Rightarrow x_2(t) < 1 \) and \( \lim_{t \to \infty} x_2 = 1 \)

For \( x_2 > -1 \), \( x_1(t) \) increases while for \( x_2 < -1 \), \( x_1(t) \) decreases. Thus solution asymptotic to \( x_2 = 1 \) as \( x_1 \to \infty \). Using

\[
\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 + 1}{x_2}.
\]

so

\[
\frac{dx_2}{dx_1} = \infty, \text{ when } x_2 = 0.
\]

Diagram of \( u^* = -1 \) paths \( C^- \). Here \( x(t) = (-Ae^{-t} - t + k, Ae^{-t} - 1) \)

**Notes:**

(i) \( A = 0 \) gives the solution \( x(t) = (-t + k, -1) \)

(ii) \( A > 0 \Rightarrow x_2(t) > -1 \) and \( \lim_{t \to \infty} x_2 = -1 \)

(iii) \( A < 0 \Rightarrow x_2(t) < -1 \) and \( \lim_{t \to \infty} x_2 = -1 \)

For \( x_2 < 1 \), \( x_1(t) \) decreases while for \( x_2 > 1 \), \( x_1(t) \) increases. Thus solution asymptotic to \( x_2 = -1 \) as \( x_1 \to -\infty \).

# To get to 0 in minimum time, the phase point \( x(t) \) must travel along a \( C^+ \) path (path with \( u^* = 1 \)) or a \( C^- \) path (path with \( u^* = -1 \)) and can switch from one to another at most once. There is only one \( C^+ \) path going to 0 which we denote by \( P^+O \). Moreover there is only one \( C^- \) path going to 0 which we denote by \( Q^-O \).
Must arrive at 0 on one of these two paths. Slope of these curves at 0 is
\[
\frac{dx_2}{dx_1} = \frac{u^*}{0} = \pm \infty
\]
Put these together:

• Initial state \( w \) on \( P^+O \) or \( Q^-O \), optimal control is \( u^* = 1 \) or \( u^* = -1 \), respectively.
• \( w \) above and to right of \( P^+OQ^- \), cannot go to \( 0 \) on a \( C^+ \) path (these go to \( P \)). So go on a \( C^- \) path until \( P^+O \) is reached, then switch
  \[
u^* = \begin{cases} 
-1 & \text{until it reaches } P^+O \\
+1 & \text{afterwards}
\end{cases}
\]
• \( w \) below and to the left of \( P^+OQ^- \), cannot go to \( 0 \) on a \( C^- \) path (these go to \( Q \)). So go on a \( C^+ \) path until \( Q^-O \), then switch to \( u^* = -1 \).
  \[
u^* = \begin{cases} 
+1 & \text{until } Q^-O \text{ reached} \\
-1 & \text{afterwards}
\end{cases}
\]

Summary.

\[
u^* = \begin{cases} 
1 & \text{below and left of } P^+OQ^- \text{ & switch on } Q^-O \\
1 & \text{on } P^+O \ldots \text{ no switch} \\
-1 & \text{above and right or } P^+OQ^- \text{ & switch on } P^+O \\
-1 & \text{on } Q^-O \ldots \text{ no switch.}
\end{cases}
\]

Question 2. Let \( \dot{x}_1 = x_2 \) and \( \dot{x}_2 = u \) where \( |u| \leq 2 \) (compare with that given in Example 2 from lectures).

(a) Find the time-optimal control from \((2, 2)\) to \((-2, 0)\) and calculate the minimum time. Find also the time-optimal control and minimum time from \((-2, 0)\) to \((2, 2)\). Comment!

(b) Suppose that the constraint is changed to \( 0 \leq t \leq 2 \). Show that there is a time-optimal control to the origin only if \( x_2 < 0 \) and \( x_1 \geq x_2^2/4 \).

Solution Question 2.

\[
\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{x} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ where } |u| \leq 2,
\]
and 
\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \]

Cost is \( t_1 \) and has to be in the form 
\[ J = \int_0^{t_1} f_0(x_1, x_2, u) \, dt \]
to use the PMP. Choose \( f_0 = 1 \). Now \( \psi_0 = -1 \), so 
\[ f_0(x_1, x_2, u) = 1, \quad f_1(x_1, x_2, u) = x_2, \quad f_2(x_1, x_2, u) = u \]  
(4)

\[
H = \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2
\]
\[ H = -1 + \psi_1 x_2 + \psi_2 u \]
\[ \dot{\psi}_1 = -\partial H/\partial x_1 = 0, \quad \dot{\psi}_2 = -\partial H/\partial x_2 = -\psi_1. \]

By PMP the optimal control \( u^* \) maximizes \( H \) as a function of \( u \). Since \( H \) is linear in \( u \) and \( |u| \leq 1 \) the maximum value of \( H \) is at \( u^* = 2 \) if \( \psi_2 > 0 \) and at \( u = -2 \) if \( \psi_2 < 0 \); that is, 
\[ u^*(t) = \text{sgn} \left( S(t) \right), \text{ where } S(t) = \psi_2(t). \]

\[ \dot{\psi}_1 = 0 \Rightarrow \psi_1 = k, \text{ where } k \text{ a constant} \]
\[ \Rightarrow \dot{\psi}_2 = -k \Rightarrow \psi_2 = -kt + l, \text{ where } l \text{ a constant.} \]
\[ \Rightarrow S = \psi_2 = l - kt \text{ has at most one zero } \Rightarrow \text{ at most one switch.} \]

Optimal trajectories satisfy 
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = u^*, \text{ where } u^* = \pm 2 \]
\[ \Rightarrow \frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{u^*}{x_2}. \]
\[ \Rightarrow x_2^2 = 2u^*x_1 + k, \text{ where } k \text{ is a constant.} \]
Diagram of $u^* = 2$ paths $C^+$. 

\[ x_2^2 = 4x_1 + k, \text{ where } k \text{ is a constant.} \]

Diagram of $u^* = -2$ paths $C^+$. 

\[ x_2^2 = -4x_1 + k, \text{ where } k \text{ is a constant.} \]

To get from (2, 2) to (-2, 0) in minimum time, the phase point $x(t)$ must travel along a $C^+$ path (path with $u^* = 2$) or a $C^-$ path (path with $u^* = -2$) and can switch from one to another at most once.

There is only one $C^-$ path going to (-2, 0) which is $x_2^2 = -2x_1 - 4$. This does not pass through (2, 2) so a solution would have to switch from the $C^+$ path through (2, 2) which is $x_2^2 = 2x_1$. These do not intersect so no solution this way.

There is only one $C^+$ path going to (-2, 0) which is $x_2^2 = 2x_1 + 4$. This does not pass through (2, 2) so a solution would have to switch from the $C^-$ path through (2, 2) which is $x_2^2 = -2x_1 + 8$. On the $C^-$ curve $\dot{x}_2 = u^* = -2$ so $x_2$ decreases while on the $C^+$ curve $\dot{x}_2 = u^* = 2$ so $x_2$ increases. These intersect at (1, $\sqrt{2}$). So this is the only solution satisfying the Pontryagin Maximum Principle.
**Question 2(b).** Now \(0 \leq u \leq 2\). From the above we see that \(u^* = 0\) if \(\psi_2 < 0\) and \(u^* = 2\) if \(\psi_2 > 0\). Thus the \(C^+\) curves are \(x_2^2 = 4x_1 + k\) and they are traversed in the direction of \(x_2\) increasing since \(\dot{x}_2 = u^* = 2\). The \(C^-\) curves are now \(x_2 = k\), a constant traversed in the direction of \(x_1\) increasing if \(x_2 > 0\) and decreasing if \(x_2 < 0\) since \(\dot{x}_1 = x_2\). Moreover \(x_2 = 0\) \(x_1 = k\) is a \(C^-\) curve for each constant \(k\).

Thus the only curve \(C^+\) through \(0 \sim\) is \(x_2^2 = 4x_1\) so \(P^+O\) is \(x_2 = k, a constant traversed in the direction of x_1 increasing if x_2 > 0 and decreasing if x_2 < 0 since \(\dot{x}_1 = x_2\). Moreover \(x_2 = 0\) \(x_1 = k\) is a \(C^-\) curve for each constant \(k\).

Thus if \(w = (a, b)\) where \(b < 0\) and \(a \geq b^2/4\).

**Question 3.** The system \(\dot{x}_1 = x_2, \dot{x}_2 = x_1 + u, |u| \leq 2\) is to be controlled from \(\sim\) \(0 \sim\) to \(\sim\) \(1 \sim\) in minimum time. Show that the time optimal control can only take the values \(+2\) or \(-2\) and that it can switch at most once. Given that \(\sim 0 = (-1, 0)\) and \(\sim 1 = (1, 0)\) show that the switch takes place at \((0, \sqrt{3})\) and find the time at which the switch takes place. Show that the minimum transfer time is \(2 \sinh^{-1} \sqrt{3}\).

**Solution.**

\[
\dot{\sim} = Ax + u \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ where } |u| \leq 2
\]

and \(A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

(1)

\[
H = -1 + \psi_1x_2 + \psi_2(x_1 + u)
\]

\[
= -1 + \psi_1x_2 + \psi_2x_1 + u\psi_2
\]

\[
\dot{\psi}_1 = -\partial H/\partial x_1 = -\psi_2, \quad \dot{\psi}_2 = -\partial H/\partial x_2 = -\psi_1, \text{ that is,}
\]

\[
\dot{\psi}_\sim = -A^T\psi = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \psi.
\]

By PMP the optimal control \(u^*\) maximizes \(H\) as a function of \(u\). Since \(H\) is linear in \(u\) and \(|u| \leq 2\) the maximum value of \(H\) is at \(u^* = 2\) if \(S > 0\) and at \(u = -2\) if \(S < 0\); that is,

\[
u^*_\sim(t) = \text{sgn } (S(t)), \text{ where } S = \psi_2.
\]

Optimal trajectories satisfy

\[
\dot{\sim} = Ax + u^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ where } u^* = \pm 2.
\]

Eigenvalues of \(A\) are \(-1, 1\) (UNSTABLE SADDLE). These are real so \(S = \psi_2(t)\) can change sign at most once \(\Rightarrow\) at most one switch.
Eigenvectors of $A$ are:

Case $\lambda_1 = -1$

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u \\ v \end{pmatrix}, \quad v = -u, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sim_1 v
\]

Case $\lambda_2 = 1$

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad v = u, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sim_2 v
\]
(3)  
# Eqlbrm point for \( u^* = 2 \)
\[
\begin{align*}
x_2 &= 0 \\
x_1 &= -2 \\
x_2 &= 0
\end{align*}
\]
= \( P \)

# Eqlbrm point for \( u^* = -2 \)
\[
\begin{align*}
x_2 &= 0 \\
x_1 &= 2 \\
x_2 &= 0
\end{align*}
\]
= \( Q \)

# To get to (1, 0) in minimum time, the phase point \( \sim \) must travel along a \( \mathcal{C}^+ \) path (path with \( u^* = 2 \)) or a \( \mathcal{C}^- \) path (path with \( u^* = -2 \)) and can switch from one to another at most once.

The \( \mathcal{C}^- \) curve through \((-1, 0)\) does not intersect the \( \mathcal{C}^+ \) curve through \((1, 0)\) so there is no solution starting at \((-1, 0)\) with \( u^* = -1 \) ⇒ look for a solution starting with \( u^* = 2 \) and switching to \( u^* = -2 \). By symmetry these must intersect on the \( x_2 \) axis at \((0, b)\) at \( t_s \), say.

There is only one \( \mathcal{C}^+ \) path starting at \((-1, 0)\). Now
\[
x = \alpha_+ \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{-t} + \beta_+ \left( \begin{array}{c} 1 \\ 1 \end{array} \right) e^t + \left( \begin{array}{c} -2 \\ 0 \end{array} \right).
\]
As \( x(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \) so \( \alpha_+ + \beta_+ - 2 = -1 \) and \(-\alpha_+ + \beta_+ = 0\), so \( \alpha_+ = \beta_+ = 1/2 \).

Moreover there is only one \( \mathcal{C}^- \) path going to \((1, 0)\) namely
\[
x = \alpha_- \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{-t} + \beta_- \left( \begin{array}{c} 1 \\ 1 \end{array} \right) e^t + \left( \begin{array}{c} 2 \\ 0 \end{array} \right).
\]

Using \( \mathcal{C}^+ \) curve passes through \((0, b)\) we have:
\[
x_1(t_s) = 0 \Rightarrow \alpha_+ e^{-t_s} + \beta_+ e^{t_s} - 2 = 0 \text{ and}
\]
\[
x_2(t_s) = b \Rightarrow -\alpha_+ e^{-t_s} + \beta_+ e^{t_s} = b. \text{ Now } \alpha_+ = \beta_+ = 1/2 \text{ so using}
\]
t _s > 0 ⇒ \( e^{t_s} > 1 \) and \( e^{-t_s} = 1/e^{t_s} \) we obtain \( e^{t_s} = 2 + \sqrt{3} \), \( e^{-t_s} = 2 - \sqrt{3} \) and \( b = \sqrt{3} \).

Using the \( \mathcal{C}^- \) curve through \((1, 0)\) also passes through \((0, b)\) at \( t = t_s \) we have:
\[
x_1(t_s) = 0 \Rightarrow -\alpha_- e^{-t_s} + \beta_- e^{t_s} + 2 = 0 \text{ and}
\]
\[
x_2(t_s) = b \Rightarrow -\alpha_- e^{-t_s} + \beta_- e^{t_s} = b. \text{ Using } e^{t_s} = 2 + \sqrt{3} \), \( e^{-t_s} = 2 - \sqrt{3} \) and \( b = \sqrt{3} \) we obtain \( \alpha_- = -(2 + \sqrt{3})^2/2 \) and \( \beta_- = -(2 - \sqrt{3})^2/2 \). Now
\[
x_2(t_1) = 0 \Rightarrow -\alpha_- e^{-t_1} + \beta_- e^{t_1} = 0 \Rightarrow (2 + \sqrt{3})^2 e^{-t_1} - (2 - \sqrt{3})^2 e^{t_1} = 0 \Rightarrow t_1/2 = \ln(2 + \sqrt{3})
\]
⇒ \( \sinh(t_1/2) = \sqrt{3} \Rightarrow t_1 = 2 \sinh^{-1} \sqrt{3} \).

**Question 4.** Solve the problem of time-optimal control to the origin for the system
\[
\dot{x}_1 = x_1 + x_2 + \alpha u, \quad \dot{x}_2 = 4x_1 + x_2 + u, \quad \text{where } |u| \leq 1
\]
in the cases (i) \( \alpha = 0 \) and (ii) \( \alpha = -1/2 \).

**Solution.**
\[
\dot{x} = A x + u \left( \begin{array}{c} \alpha \\ 1 \end{array} \right), \quad \text{where } |u| \leq 1
\]
and \( A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \).

(1)

\[
H = -1 + \psi_1(x_1 + x_2 + \alpha u) + \psi_2(4x_1 + x_2 + u)
\]

\[
= -1 + \psi_1(x_1 + x_2) + \psi_2(4x_1 + x_2) + u(\alpha \psi_1 + \psi_2)
\]

\[
\dot{\psi}_1 = -\partial H/\partial x_1 = -\psi_1 - 4\psi_2; \quad \dot{\psi}_2 = -\partial H/\partial x_2 = -\psi_1 - \psi_2, \text{ that is,}
\]

\[
\dot{\psi} = -A^T \psi = \begin{pmatrix} -1 & -4 \\ -1 & -1 \end{pmatrix} \psi.
\]

By PMP the optimal control \( u^* \) maximizes \( H \) as a function of \( u \). Since \( H \) is linear in \( u \) and \( |u| \leq 1 \) the maximum value of \( H \) is at \( u^* = 1 \) if \( S > 0 \) and at \( u = -1 \) if \( S < 0 \); that is,

\[
u^*(t) = \text{sgn } (S(t)), \text{ where } S = \alpha \psi_1 + \psi_2.
\]

Optimal trajectories satisfy

\[
\dot{x} = Ax + u^* \begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \quad u^* = \pm 1.
\]

Eigenvalues of \( A \) are -1, 3 (UNSTABLE SADDLE). These are real so \( S(t) \) can change sign at most once \( \Rightarrow \) at most one switch.

# Eigenvectors of \( A \) are:

Case \( \lambda_1 = -1 \)

\[
\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u \\ v \end{pmatrix}, \quad v = -2u, \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix} = v_1
\]

Case \( \lambda_2 = 3 \)

\[
\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 3\begin{pmatrix} u \\ v \end{pmatrix}, \quad v = 2u, \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} = v_2
\]
To get to 0 in minimum time, the phase point \( x(\sim) \) must travel along a \( C^+ \) path (path with \( u^* = 1 \)) or a \( C^- \) path (path with \( u^* = -1 \)) and can switch from one to another at most once. There is only one \( C^+ \) path going to 0 which we denote by \( P^+O \). Moreover there is only one \( C^- \) path going to 0 which we denote by \( Q^-O \). Must arrive at 0 on one of these two paths. Slope of these curves at 0 is

\[
\frac{dx_2}{dx_1} = \frac{1}{\alpha} = \begin{cases} 
-2, & \text{if } \alpha = -1/2 \\
\infty, & \text{if } \alpha = 0.
\end{cases}
\]

Put these together: Case \( \alpha = -1/2 \).

- Initial state \( w \) on \( P^+O \) or \( Q^-O \), optimal control is \( u^* = 1 \) or \( u^* = -1 \), respectively.
- \( w \) not on \( P^+OQ^- \). Since \( P^+OQ^- \) is part of a solution for both \( u^* = 1 \) and \( u^* = -1 \) cannot switch to \( P^+OQ^- \) from either a \( C^- \) path or a \( C^+ \) path. Thus cannot control to the origin with at most one switch so no optimal control.

Put these together: Case \( \alpha = 0 \).

- Initial state \( w \) on \( P^+O \) or \( Q^-O \), optimal control is \( u^* = 1 \) or \( u^* = -1 \), respectively.
• Case $w$ not on $P^+OQ^-$.  
Let APB be the line through P parallel to the direction $v$ and CQD be the line through Q parallel to the direction $v$. Let $D$ be the region inside APB and CQD.

Summary.

$$u^* = \begin{cases} 
-1 & \text{on } Q^-O, \text{ no switch} \\
-1 & \text{inside } D \text{ below to left of } P^+OQ^- \text{ and switch when hit } P^+O \\
1 & \text{inside } D \text{ above to right of } P^+OQ^- \text{ and switch when hit } Q^-O \\
1 & \text{on } P^+O, \text{ no switch.} 
\end{cases}$$

It is not controllable to the origin from on the boundary or from outside $D$.

Justification:

Let the initial state $w$ be to the left and below APB. Solutions corresponding to $u^* = 1$ cannot cross APB and so cannot switch to $Q^-O$ and be controlled to the origin with at most one switch.

Solutions corresponding to $u^* = -1$ move away from APB and so cannot cross APB and so cannot switch to $Q^-O$ and be controlled to the origin with at most one switch.

Similarly if the initial state $w$ be to the right and above CQD the solution cannot be controlled to the origin with at most one switch.