MATH3404 Tutorial Sheet 9 (week 11)

Question 1. Solve the problem of time-optimal control to the origin for each of the following systems

(i) \( \dot{x}_1 = -3x_1 + 2x_2, \quad \dot{x}_2 = 2x_1 - 3x_2 + u, \quad \text{where } |u| \leq 1 \)
(ii) \( \dot{x}_1 = x_2, \quad x_2 = -3x_1 - 4x_2 + u, \quad \text{where } |u| \leq 1 \)
(iii) \( \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u, \quad \text{where } |u| \leq 1 \)

Question 2*. Let \( \dot{x}_1 = x_2 \) and \( \dot{x}_2 = u \) where \( |u| \leq 2 \) (compare with that given in Example 2 from lectures).

(a) Find the time-optimal control from \((2, 2)\) to \((-2, 0)\) and calculate the minimum time. Find also the time-optimal control and minimum time from \((-2, 0)\) to \((2, 2)\). Comment!

(b) Suppose that the constraint is changed to \(0 \leq t \leq 2\). Show that there is a time-optimal control to the origin only if \(x_2 < 0\) and \(x_1 \geq \frac{x_2^2}{4}\).

Question 3. The system \( \dot{x}_1 = x_2, \dot{x}_2 = x_1 + u \), \( |u| \leq 2 \) is to be controlled from \( x^0 \) to \( x^1 \) in minimum time. Show that the time optimal control can only take the values +2 or −2 and that it can switch at most once. Given that \( x^0 = (-1, 0) \) and \( x^1 = (1, 0) \) show that the switch takes place at \((0, \sqrt{3})\) and find the time at which the switch takes place. Show that the minimum transfer time is \( 2 \sinh^{-1} \sqrt{3} \).

Question 4. Solve the problem of time-optimal control to the origin for the system

\[ \dot{x}_1 = x_1 + x_2 + \alpha u, \quad \dot{x}_2 = 4x_1 + x_2 + u, \quad \text{where } |u| \leq 1 \quad (1) \]

in the cases (i) \( \alpha = 0 \) and (ii) \( \alpha = -1/2 \).