**MATH3404 Tutorial Sheet 8 (week 9)**

1*. Let \( \dot{x}_1 = x_2 = f_1 \) and \( \dot{x}_2 = u = f_2 \) where \( |u| \leq K/m \). Suppose that the initial state is \( x^0 = (x^0_1, x^0_2) = (a, b) \). Control the system to the origin in minimum time. Suppose the initial state \((a, b)\) lies above the switching curve POQ and \( t = \eta \) is the time at which the optimal control switches from \(-K/m\) to \( K/m\).

(i) Show that at \( t = \eta \) the system is at \((x^0_1, x^0_2) = (l, s)\) where
\[
l = a + m(b^2 + 2Ka/m)/(4K), \quad s = -(2Kl/m)^{1/2}
\]  

(ii) Apply the condition \( H = 0 \) at \( t = 0, \ t = \eta \) and \( t = t_1 \) and deduce that
\[
A = 1/s \quad B = m(b - s)/(Ks), \quad \eta = m(b - s)/K, \ t_1 = m(b - 2s)/K
\]  

(iii) Calculate \( H \) as a function of \( t \) in the two time intervals \([0, \eta]\) and \([\eta, t_1]\). Hence verify that \( H = 0 \) for all \( t \) in \([0, t_1]\).

2*. The system \( \dot{x}_1 = -x_1 + u \), where \( u = u(t) \) is not subject to any constraint, is to be controlled from \( x_1(0) = 1 \) to \( x_1(t_1) = 2 \) where \( t_1 \) is unspecified, in such a way that
\[
J = \frac{1}{2} \int_0^{t_1} (x_1^2 + u^2) \, dt
\]
is minimized. Find the optimal control.

3. The system \( \dot{x}_1 = x_1 + u \), where \( u = u(t) \) is not subject to any constraint, is to be controlled from \( x_1(0) = a \) to \( x_1(t_1) = b \) where \( t_1 \) is unspecified, in such a way that
\[
J = \frac{1}{2} \int_0^{t_1} (2x_1^2 + 2ux_1 + u^2) \, dt
\]
is minimized. Find the optimal control.