MATH3404 TUTORIAL SHEET 7 (Week 8)

Question 1.

(A) Let*

$$J(x) = \int_{t_0}^{t_1} \left(s(t)\dot{x}^2 + q(t)x^2 \right) dt,$$

where $s(t)$ and $q(t)$ are C^1. Show that the extremals of J must satisfy

$$\frac{d}{dt}\left(s(t)\dot{x} \right) = q(t)x. \qquad (1)$$

Show that if $s(t)$ and $q(t)$ are positive on $[t_0, t_1]$ with $x(t_0) = x_0$ and $x(t_1) = x_1$, then any curve $x = \phi(t)$ satisfying equation (1) and passing through the end - points gives J a strong local minimum by the Weierstrass's sufficient Theorem. Show that the minimum value of J is

$$J_{\min} = [s(t)\phi(t)\dot{\phi}(t)]_{t_0}^{t_1}.$$

(B) to find the minimizing curve and the value of $J_{\min}$ for the cases

$$\int_1^2 \left(\dot{x}^2 + \frac{2}{t^2}x^2 \right) dt, \text{ where } x(1) = 1,\ x(2) = 4,$$

(This is a very challenging question.)

Question 2. *Let Ω be a bounded open domain. Suppose that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and consider the graph of the function u*

$$G_u = \{(x, y, u(x, y)) : (x, y) \in \Omega\}.$$

Then the area of the surface by the graph is

$$Area(G_u) = \int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dxdy = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dxdy$$

If the graph G_u is minimizing, you verify that u satisfies the following the Euler-Lagrange equation:

$$div \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

The equation can be written as

$$0 = (1 + u_y^2)u_{xx} + (1 + u_x^2)u_u u - 2u_x u_y u_{xy}.$$