Costate equations

$$\dot{\psi}_1 = -\frac{\partial H}{\partial x_1} = 3x_1 + \psi_1$$

(Optimal) State equation $\dot{x}_1 = -x_1 + \psi_1$

$$\begin{aligned}
\ddot{x}_1 &= -\dot{x}_1 + \dot{\psi}_1 = x_1 - \psi_1 + 3x_1 + \psi_1 \\
&= 4x_1 \\
\Rightarrow \quad x_1 &= Ae^{2t} + Be^{-2t} \\
\psi_1 &= \dot{x}_1 + x_1 \\
&= 2Ae^{2t} - 2Be^{-2t} + Ae^{2t} + Be^{-2t} \\
&= 3Ae^{2t} - Be^{-2t} \\
&= u^*.
\end{aligned}$$

End conditons:

$$\begin{aligned}
x_1 &= 0 \qquad \text{at } t = 0 \\
x_1 &= 2 \qquad \text{at } t = 1
\end{aligned}$$

$$
\begin{aligned}
0 &= A + B \qquad \Rightarrow \qquad B = -A \\
2 &= Ae^2 + Be^{-2} = A(e^2 - e^{-2}) = 2A \sinh 2 \\
A &= 1/\sinh 2.
\end{aligned}
$$

Optimal control is

$$
u^* = \frac{1}{\sinh 2}(3e^{2t} + e^{-2t})
$$

(Considerably simpler because we knew t_1).

Time Optimal Control of Linear Systems

$|u(t)| \leq 1$, piecewise continuous.

#

$$
\left.\begin{aligned}
\dot{x}_1 &= ax_1 + bx_2 + lu \\
\dot{x}_2 &= cx_1 + dx_2 + mu
\end{aligned}\right\} \quad \begin{array}{l} a, b, c, d, \\ l, m \\ \text{constant} \end{array}
$$

$$
\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{l}u \qquad \underset{\sim}{l} = \begin{pmatrix} l \\ m \end{pmatrix}
$$

Control system from $\underset{\sim}{x}(t_0) = \underset{\sim}{x}^0$ to $\underset{\sim}{x}(t_1) = \underset{\sim}{x}^1$ by

an admissible control, minimizing

$$J = \int_{t_0}^{t_1} 1\,dt = t_1 - t_0. \quad (f_0 = 1)$$

Solutions will involve PHASE PLANE ANALYSIS.

$$\det(A - \lambda I) = 0.$$

Revision:

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x}\ , \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ , \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution

$$\underset{\sim}{x} = \alpha_1 \underset{\sim 1}{v} e^{\lambda_1 t} + \alpha_2 \underset{\sim 2}{v} e^{\lambda_2 t},$$

where $\lambda_1,\ \lambda_2$ are eigenvalues of A and $\underset{\sim 1}{v}, \underset{\sim 2}{v}$ are the corresponding eigenvectors.

Case 1. Real nonzero eigenvalues or same sign.

(a) $\lambda_2 < \lambda_1 < 0\,;$ (b) $0 < \lambda_1 < \lambda_2.$

STABLE NODE UNSTABLE NODE

Case 2. Real, nonzero eigenvalues, of opposite sign

$\lambda_1 < 0 < \lambda_2$

Saddle,

Unstable

Will do Case 3, imaginary and complex eigenvalues later if time permits

Return to $\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{l}\,u\ , \quad f_0 = 1$

(1)

$$\begin{aligned} H &= \sum \psi_i f_i \\ &= -1 + \psi_1(ax_1 + bx_2 + lu) \\ &\qquad +\psi_2(cx_1 + dx_2 + mu) \\ &= -1 + \psi_1(ax_1 + bx_2) + \psi_2(cx_1 + dx_2) \\ &\qquad +u(l\psi_1 + m\psi_2) \\ \dot{\psi}_1 &= -a\psi_1 - c\psi_2 \\ \dot{\psi}_2 &= -b\psi_1 - d\psi_2 \\ \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \dot{\underset{\sim}{\psi}} &= -A^T \underset{\sim}{\psi}. \end{aligned}$$

(2) MAXIMIZE H (as a function of u)

Linear in u, so $u^* = \pm 1$, depending on the sign of $l\psi_1 + m\psi_2$:

$$u^* = sgn(l\psi_1 + m\psi_2).$$

Piecewise continuous controls, switch when $l\psi_1 + m\psi_2$ changes sign (i.e. at zeros of $l\psi_1 + m\psi_2$)

$$S = l\psi_1(t) + m\psi_2(t)$$

SWITCHING FUNCTION.

Between any two adjacent zeros of S, u^* is constant:

State Equation $\quad \dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{l}\,u^*, \quad u^* = +1 \text{ or } -1.$

If $\det A \neq 0$, trajectories for $u^* = 1$ have an isolated singularity (equilibrium points) at the solution of

$$\begin{aligned} ax_1 + bx_2 + l &= 0 \\ cx_1 + dx_2 + m &= 0. \end{aligned}$$

For $u^* = -1$, the equilibrium is at the solution of

$$\begin{aligned} ax_1 + bx_2 - l &= 0 \\ cx_1 + dx_2 - m &= 0. \end{aligned}$$

Behaviour of both families of trajectories is governed by the eigenvalues of the "system matrix" A. The pattern is the same as that of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, translated to the equilibrium point.

If A has real eigenvalues, $-A^T$ has real eigenvalues also:

$$
\begin{aligned}
A: \qquad & \lambda^2 - (a+d)\lambda + \det A = 0 \\
A^T: \qquad & \lambda^2 + (a+d)\lambda + \det A = 0.
\end{aligned}
$$

Solution of costate equations has the form

$$\underset{\sim}{\psi} = \underset{\sim}{h} e^{q_1 t} + \underset{\sim}{k} e^{q_2 t}$$

q_1, q_2 eigenvalues of $-A^T$

$\underset{\sim}{h}$, $\underset{\sim}{k}$ corresponding eigenvectors.

Hence, switching function is of the form

$$
\begin{aligned}
S &= l\psi_1 + m\psi_2 \\
&= Le^{q_1 t} + Me^{q_2 t}
\end{aligned}
$$

has at most one zero.

Lemma. (i) If eigenvalues of A are real, the switching function has at most one zero.

(ii) Only possible optimal control sequences are

$u^* = 1\,, \quad t_0 \le t \le t_1$

$u^* = -1\,, \quad t_0 \le t \le t_1$

$u^* = \begin{cases} +1\,, & t_0 \le t \le \tau \\ -1\,, & \tau \le t \le t_1 \end{cases}$ τ is a zero of S.

$u^* = \begin{cases} -1\,, & t_0 \le t \le \tau \\ +1\,, & \tau \le t \le t_1 \end{cases}$

Remark. It can be shown that PMP is both necessary and sufficient for linear time-optimal control problems. Hence, if we can find a control sequence of the type on the lemma, it must be an optimal control.

Example.

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 2x_2 + 5u \\ \dot{x}_2 &= 2x_1 - 3x_2 \end{aligned}$$

to control from any initial state to the origin in minimum time, $|u| \le 1$. Find the optimal control u^* minimizing

$$J = \int_{t_0}^{t_1} 1\,dt = t_1 - t_0$$

Solution.

$$f_0 = 1\,, \quad \begin{aligned} f_1 &= -3x_1 + 2x_2 + 5u \\ f_2 &= 2x_1 - 3x_2 \end{aligned}$$

(1)

$$\begin{aligned} H &= -1 + \psi_1(-3x_1 + 2x_2 + 5u) \\ &\quad +\psi_2(2x_1 - 3x_2) \\ &= -1 + \psi_1(-3x_1 + ex_2) + \psi_2(2x_1 - 3x_2) \\ &\quad +u(5\psi_1) \end{aligned}$$

$$\begin{aligned} \dot\psi_1 &= -\tfrac{\partial H}{\partial x_1} = 3\psi_1 - 2\psi_2 \\ \dot\psi_2 &= -\tfrac{\partial H}{\partial x_2} = -2\psi_1 + 3\psi_2 \end{aligned} \qquad \dot{\underset{\sim}{\psi}} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \underset{\sim}{\psi}$$

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{l}\,u\,, \quad A = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}\,, \quad \underset{\sim}{l} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\dot{\underset{\sim}{\psi}} = -A^T\psi$$

(2) H is maximized (wrt u) when

$$u^* = sgn(5\psi_1) = \pm 1$$

(3) Optimal solutions of the state eqns are given by

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} u^* \ , \ u^* = \pm 1$$

Eigenvalues of

$$A \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = (3+\lambda)^2 - 4$$

$$= \lambda^2 + 6\lambda + 5 = 0$$

$\lambda = -1, -5.$

The lemma means that there is at most one switch.

Since both eigenvalues are negative, the trajectories are a *stable node* at the equilibrium point.

$u^* = 1$, eqlbrm given by

$$\left.\begin{aligned} -3x_1 + 2x_2 + 5 &= 0 \\ 2x_1 - 3x_2 &= 0 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

\# $u^* = -1$, eqlbrm:

$$\left.\begin{aligned} -3x_1 + 2x_2 - 5 &= 0 \\ 2x_1 - 3x_2 &= 0 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= -3 \\ x_2 &= -2 \end{aligned}$$

\# Eigenvectors of A are:

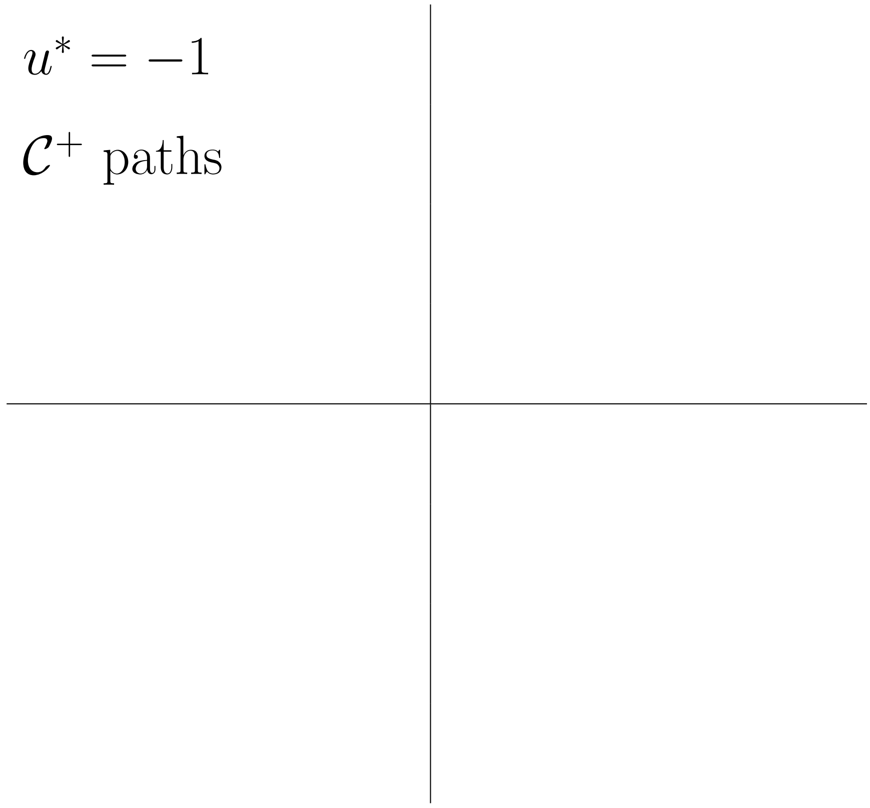
$$\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u \\ v \end{pmatrix}, \quad v = u\,, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underset{\sim}{v}_1$$

$$\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -5\begin{pmatrix} u \\ v \end{pmatrix}, v = -u, \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underset{\sim}{v}_2$$

$u^* = 1$

$\mathcal{C}^+$ paths

$$\underset{\sim}{x}(t) = \alpha \underset{\sim}{v}_1 e^{-t} + \beta \underset{\sim}{v}_2 e^{-5t} + \binom{3}{2}$$



\# To get to $\underset{\sim}{0}$ in minimum time, the phase point $\underset{\sim}{x}(t)$ must travel along a $\mathcal{C}^+$ path or a $\mathcal{C}^-$ path and can switch from one to another at most once. There is only one $\mathcal{C}^-$ path going to $\underset{\sim}{0}$.

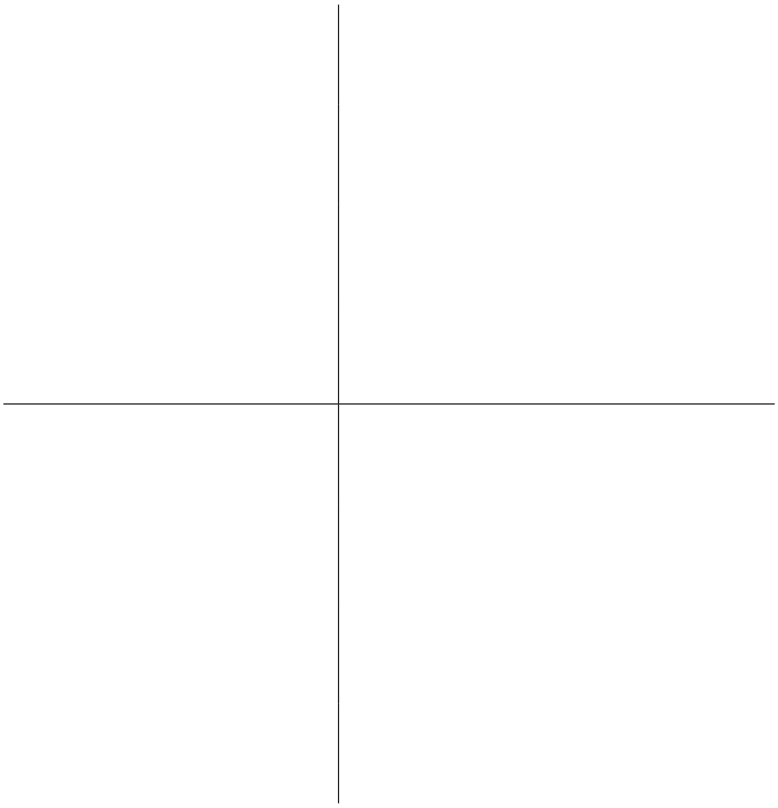
\# Must arrive at $\underset{\sim}{0}$

on these two paths.

Slope of these curves at $\underset{\sim}{0}$ is

$$\frac{dx_2}{dx_1} = \frac{0}{5u^*} = 0$$

Put these together:



- Initial state w on P^+O or Q^-O, optimal control is $u^* = 1$ or $u^* = -1$ respectively.

- W above P^+OQ^-, cannot go to 0 on a $\mathcal{C}^+$ path (these go to P). So go on a $\mathcal{C}^-$ path until P^+O is reached, then switch

$$u^* = \begin{cases} -1 & \text{until it reaches } P^+O \\ +1 & \text{afterwards} \end{cases}$$

- W below P^+OQ^-, cannot go to 0 on a $\mathcal{C}^-$ path (these go to Q). So to on a $\mathcal{C}^+$ path until Q^-O, then switch to $u^* = -1$.

$$u^* = \begin{cases} +1 & \text{until } Q^-O \text{ reached} \\ -1 & \text{afterwards} \end{cases}$$

Summary.

$$u^* = \begin{cases} 1 & \text{below } P^+OQ^- \\ & \quad\quad \& \text{ on } P^+O \\ -1 & \text{above } P^+OQ^- \\ & \quad\quad \& \text{ on } Q^-O. \end{cases}$$

Example 1.

LAST

Stable Node

Find Eigenvectors

Find EQLBRM for
- $u^* = \ \ 1,\ P;\ \mathcal{C}^+$
- $u^* = -1,\ Q;\ \mathcal{C}^-$

On $\mathcal{C}^+$ curve $\to \underset{\sim}{0}$, P^+O

On $\mathcal{C}^-$ curve $\to 0$, Q^-O

Above P^+OQ^-:
- $\mathcal{C}^+$ curve $\to P$
- $\mathcal{C}^-$ curve $\to P^+O$

Below P^+OQ^-:
- $\mathcal{C}^-$ curve $\to Q$
- $\mathcal{C}^+$ curve $\to Q^-O$

Optimal Control

$$u^* = \begin{cases} 1 \text{ below } P^+OQ^- \text{ \& on } P^+O \\ -1 \text{ above } P^+OQ^- \\ \qquad \text{\& on } Q^-O \end{cases}$$

Example 2.

$$\begin{aligned} \dot{x}_1 &= 3x_1 + 2x_2 + 5u \\ \dot{x}_2 &= 2x_1 + 3x_2, \qquad |u| \le 1 \end{aligned}$$

Control to $\underset{\sim}{0}$ in minimum time.

Solution.

$$\underset{\sim}{\dot{x}} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \underset{\sim}{x} + u \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

(1)

$$\begin{aligned} H &= -1 + \psi_1(3x_1 + 2x_2 + 5u) + \psi_2(2x_1 + 3x_2) \\ &= -1 + \psi_1(3x_1 + 2x_2) + \psi_2(2x_1 + 3x_2) + u(5\psi_1) \\ \underset{\sim}{\dot{\psi}} &= -A^T \underset{\sim}{\psi} = \begin{pmatrix} -3 & -2 \\ -2 & -3 \end{pmatrix} \underset{\sim}{\psi}. \end{aligned}$$

(2) H is linear in u, $|u| \leq 1$

$\Rightarrow H$ max at $u^* = sgn(5\psi_1) = \pm 1$

Optimal trajectories satisfy

$$\underset{\sim}{\dot{x}} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} x + u^* \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \; u^* = \pm 1$$

Eigenvalues of A are 1, 5 UNSTABLE NODE

(3)

Eqlbrm point for $u^* = 1$

$$\left.\begin{aligned} 3x_1 + 2x_2 + 5 = 0 \\ 2x_1 + 3x_2 = 0 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= -3 \\ x_2 &= +2 \end{aligned}$$

Eqn for $u^* = -1$

$$\left.\begin{aligned} 3x_1 + 2x_2 - 5 = 0 \\ 2x_1 + 3x_2 = 0 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= 3 \\ x_2 &= -2 \end{aligned}$$

(4) Try to construct time-optimal paths by finding routes to $\underset{\sim}{0}$ (which we can do, even though P, Q are unstable)

$\mathcal{C}^+$: only states lying on PO get to $\underset{\sim}{0}$. If we choose $u^* = 1$, the $\mathcal{C}^+$ curve through W (outside controllability region) cannot meet QO and cannot control to $\underset{\sim}{0}$.

$\mathcal{C}^-$: control to $\underset{\sim}{0}$ has to intersect PO, cannot do this for V outside control region.

For most initial states, the trajectory will not get to $\underset{\sim}{0}$ under any control.

Small finite region of controllability.

Larger K, $|u| \le K$, larger region of controllability.

Example 3.

$$\begin{aligned}\dot{x}_1 &= x_1 + 3x_2 - 7u \\ \dot{x}_2 &= 3x_1 + x_2 - 5u\end{aligned}\quad , \quad |u| \le 1$$

control to origin in minimum time.

Solution.

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} -7 \\ -5 \end{pmatrix} u$$

(1)

$$\begin{aligned} H = -1 + \psi_1(x_1 + 3x_2) + \psi_2(3x_1 + x_2) \\ +u(-7\psi_1 - 5\psi_2) \end{aligned}$$

(2) Maximized (for $|u| \le 1$) if

$$u^* = sgn(-7\psi_1 - 5\psi_2) = \pm 1.$$

Optimal trajectories satisfy

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} x + u^* \begin{pmatrix} -7 \\ -5 \end{pmatrix} , \quad u^* = \pm 1$$

(3) Eigenvalues of A:

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = -2,\ 4 \text{ saddle point (unstable)}$$

Eigenvectors: $\lambda = -2$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 \\ -2v_2 \end{pmatrix}$$

$$\lambda = -2 \qquad v_1 + 3v_2 = -2v_1,\ v_2 = -v\,;\ \binom{1}{-1} = \underset{\sim 1}{v}$$

Eigenvectors: $\lambda = 4$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \end{pmatrix}$$

$$\lambda = 4 \qquad v_1 + 3v_2 = 4v_1,\ v_2 = v_1; \quad \binom{1}{1} = \underset{\sim 2}{v}$$

(4) **Eqlbrm** at $u^* = 1$,

$$x_1 + 3x_2 - 7 = 0$$

$$3x_1 + x_2 - 5 = 0$$

$$3x_1 + 9x_2 - 21 = 0$$

$$8x_2 = 16\,, \quad \left.\begin{aligned} x_2 &= 2 \\ x_1 &= 1 \end{aligned}\right\} \quad P$$

Eqlbrm at $u^* = -1$

$$x_1 + 3x_2 + 7 = 0$$

$$\left.\begin{aligned} 3x_1 + 2x_2 + 5 &= 0 \\ 3x_1 + 9x_2 + 21 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} x_2 &= -2 \\ x_1 &= -1 \end{aligned}\right\} Q$$

Region of controllability is an infinite strip. Outside the strip, control to $\underset{\sim}{0}$ is impossible. Inside,

$$u^* = \begin{cases} -1 & \text{below } Q^- O P^+ \text{ \& on } Q^- O \\ +1 & \text{above } Q^- O P^+ \text{ \& on } O P^+. \end{cases}$$