So the minimum time path is constructed as follows:

Optimal control at each t must be K/m or $-K/m$;

Control switches from one value to the other AT MOST ONCE.

To get to the origin, go until the trajectory cuts one of the parabolic arcs heading towards $\underset{\sim}{0}$; these are the only routes to $\underset{\sim}{0}$.

Optimal Control

$$u^*(t) = \begin{cases} \frac{-K}{m} & \text{above POQ} \\ & \quad \text{and on OP} \\ +\frac{K}{m} & \text{below POQ} \\ & \quad \text{and on OQ} \end{cases}$$

POQ is the switching curve.

The minimum time of transfer t_1 could be determined by the condition $H = 0$ (on optimal trajectory) for all t. There are 3 unknowns A, B and t_1. Compute $H(t)$ at $t = 0$, $t =$ switch time η, $t = t_1$ (see Qn.1 on Tute sheet). However, can avoid calculating A and B, just finding t_1, directly using the state equations. We consider the case (a, b) lies above the switching curve POQ.

$$\begin{aligned} \dot{x}_2 = -K/m \Rightarrow x_2 &= -\frac{K}{m}t + \text{ const.} \\ &= -\frac{K}{m}t + b, \quad 0 \le t \le \eta, \ (1) \end{aligned}$$

since $x_2(0) = b$. At $t = \eta$, $x_2 = s \Rightarrow$

$$\eta = m(b - s)/K \tag{2}$$

by (1), where s is undetermined. For $t > \eta$,

$$\dot{x}_2 = K/m \Rightarrow x_2 = \frac{K}{m}t + \text{ const.} \tag{3}$$

Now $x_2 = s$ at $t = \eta$ so

$$x_2 = Kt/m - b + 2s \tag{4}$$

by (2) and (3). At $t = t_1$, $x_2 = 0$, so $t_1 = m(b-2s)/K$ by (4). Then s is determined by finding where the path through (a, b) intersects PO. Note that on PO $x_2^2 = 2(K/m)x_1$ so if we switch on PO at $t = \eta$, $x_2 = s$ then $x_1 = ms^2/(2K)$. Now $\dot{x}_1 = x_2 = -\frac{K}{m}t + b$ for $0 \le t \le \eta$, so $x_1 = -\frac{K}{2m}t^2 + bt +$ const $= -\frac{K}{2m}t^2 + bt + a$ since $x_1 = a$ at $t = 0$. At $t = \eta$, $x_1 = ms^2/(2K)$ so $ms^2/(2K) = -\frac{K}{2m}\eta^2 + b\eta + a$ but $\eta = m(b - s)/K$ so we can solve for s.

Example 3. Glucose problem:

$$\dot{x}_1 = -\alpha x_1 + u, \quad 0 \leq u \leq m,$$

controlled from $x_1 = a$ at $t = 0$ to $x_1 = c$ at some time T such that

$$J = \int_0^T u\,dt$$

is minimized.

Solution. Assume a, $c \geq 0$ and $m \geq \alpha c$, otherwise the system is not controllable.

$$H = -u + \psi_1(-\alpha x_1 + u) = -\alpha x_1 \psi_1 + u(\psi_1 - 1).$$

Since H is linear in u, max of H with respect to u is for

$$u = \bar{u} = \begin{cases} 0\,, & \text{when } \psi_1 < 1 \\ m\,, & \text{when } \psi_1 > 1. \end{cases}$$

The costate variable satisfies

$$\dot{\psi}_1 = -\partial H/\partial x_1 = \alpha \psi_1 \Rightarrow \psi_1 = Ae^{\alpha t}.$$

Since $\alpha > 0$ it follows that $e^{\alpha t} > 0$ for $t > 0$, so the *switching function* (coefficient of u in H,)

$$\psi_1 - 1 = Ae^{\alpha t} - 1$$

can only have a zero in $t > 0$ if $0 < A < 1$. For $A \geq 1$, it is > 0, $\forall t > 0$, giving $\bar{u} = m$; for $A \leq 0$, it is < 0 $\forall t > 0$, giving $\bar{u} = 0$. In all 3 cases, the control maximizing H is piecewise constant. So at $t = 0$, we have either $\bar{u} = 0$ or $\bar{u} = m$.

There is no switch which can be seen as follows:

$\# \ u(0) = 0, \quad H_{t=0} = -\alpha\, a\, A = 0 \Rightarrow A = 0$

and $\psi_1 - 1 = -1 \qquad \forall t > 0$

$\# \ u(0) = m$; at $t = 0$, $H = -m + A(m - \alpha a) = 0$

$$\Rightarrow \quad A = m/(m - \alpha a).$$

Note $m - \alpha = 0$ leads to the contradiction $m = 0$.

Thus either $m < \alpha a$ (and thus $A < 0$)

or $m > \alpha a$ (and thus $A > 1$).

In either case there is no switch.

Hence $\overline{u} = 0\ \forall t$, or $\overline{u} = m \quad \forall t$. Eqn

$$\dot{x}_1 = -\alpha_1 x_1 + \bar{u}$$

integrates to

$$x_1 = Be^{-\alpha t} + \bar{u}/\alpha.$$

End conditions: $x_1(0) = a$, $x_1(T) = c$ give

$$a = B + \bar{u}/\alpha\,, \quad c = Be^{-\alpha T} + \bar{u}/\alpha$$

So

$$\begin{aligned} B &= a - \bar{u}/\alpha \\ T &= \frac{1}{\alpha}\ln\left(\frac{\bar{u} - \alpha a}{\bar{u} - \alpha c}\right). \end{aligned}$$

Case:

$a > c$, $\bar{u} = 0$ and exponential decay to c, $T = \frac{1}{\alpha}\ln\frac{a}{c}$,

$J = 0$.

$a < c$, $\bar{u} = m$ until glucose level increases to c:

$$\begin{aligned} T &= \frac{1}{\alpha}\ln\left(\frac{m-\alpha a}{m-\alpha c}\right) \\ J &= \frac{m}{\alpha}\ln\left(\frac{m-\alpha a}{m-\alpha c}\right). \\ &= \int_0^T u\,dt = \int_0^T m\,dt = mT. \end{aligned}$$

Time of Arrival Fixed

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, u) \end{aligned}$$

to be controlled from $\underset{\sim}{x}^0$ at $t = t_0$ to $\underset{\sim}{x}^1$ at $t = t_1$, where t_1 is *fixed* and *known*, so as to minimize

$$J = \int_{t_0}^{t_1} f_0(x_1, x_2, u)dt.$$

Find the optimal control.

In previous examples, we needed "$H = 0$" + end-point conditions to determine arbitrary constants (from solving DE) and t_1. But here we know t_1 already, and so the endpoint conditions are sufficient to solve the problem. In fact $H \equiv C$, a constant and $C \neq 0$ is a possibility.

Example 4. $\dot{x}_1 = -x_1 + u$ to be controlled from $x_1 = 0$ at $t = 0$ to $x_1 = 2$ at $t = 1$, minimizing

$$J = \frac{1}{2}\int_0^1 (3x_1^2 + u^2)dt$$

(no constraint on $u(t)$). Find the optimal control.

Solution. Observe that t_1 is known. Take $\psi_0 = -1$. Then

$$\begin{aligned} H &= \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2 \\ &= -\frac{1}{2}(3x_1^2 + u^2) + \psi_1(-x_1 + u). \end{aligned}$$

No constraint on u, we maximinize H by considering

$$\begin{aligned} 0 = \partial H/\partial u &= -u + \psi_1 \Rightarrow u = \psi_1 \\ \partial^2 H/\partial u^2 &= -1 < 0, \text{ so a maximum.} \end{aligned}$$