OPTIMAL CONTROL

Recall: Isoperimetric Problems included constraints. In control theory, functionals are minimized (or maximized) subject to:

constraints which are $D\,E\,s$;

the ADMISSIBLE FUNCTIONS are less well-behaved;

solutions satisfy necessary conditions which generalise the Euler-Lagrange equation.

Sample Problem 1: Level of glucose in bloodstream $x(t)$ is metabolised at a rate $\alpha x(t)$. Change (control) the level of x from $x = a$ at time 0 to $x = c$ at a time T, by infusing at a rate $u(t)$, while minimizing the total amount given to a patient,

$$J = \int_0^T u(t)dt.$$

That is, minimize $J = \int_0^T u\, dt$, subject to

$$\dot{x} = -\alpha x + u(t).$$

SAMPLE PROBLEM 2: A particle moves in the plane, force $\underset{\sim}{F}(t) = \binom{u_1}{u_2}$ (per unit mass)

$$\ddot{x} = u_1\,, \qquad \ddot{y} = u_2.$$

This gives a set of four, 1st order $D\,E\,s$ by

$$\begin{aligned} x_1 &= x, \quad \dot{x}_1 = x_2, \quad x_3 = y, \quad \dot{x}_3 = x_4 \\ \dot{x}_1 &= x_2, \quad \dot{x}_2 = u_1 \qquad\qquad\qquad (*) \\ \dot{x}_3 &= x_4, \quad \dot{x}_4 = u_2. \end{aligned}$$

STATE SPACE of the system is given by the vectors $\underset{\sim}{x} = (x_1, x_2, x_3, x_4)^T$; here the ^{T} stands for the transpose of the vector so $\underset{\sim}{x}$ is a column matrix. The state space variables specify the position and velocity of the system at t. The quantities $\underset{\sim}{u} = \binom{u_1}{u_2}$ are *controls*. Realistically, $\left|\underset{\sim}{F}\right| \leq K$, or $|u_i| \leq 1$, $i = 1, 2$, or $\underset{\sim}{u} \in U$ for some such subregion, U, in *control space*.

Control Problem: Given the DE (∗), control the system from an initial state $\underset{\sim}{x}(t_0) = \underset{\sim 0}{x}$ to a given **TARGET STATE** $\underset{\sim}{x}(t_1) = \underset{\sim 1}{x}$ (t_1 usually free). If this is possible, say the system is CONTROLLABLE. If a system is controllable from $\underset{\sim 0}{x}$ to $\underset{\sim 1}{x}$ by $\underset{\sim}{u} \in U$, find the control $\underset{\sim}{u} = \underset{\sim}{u}(t)$ minimizing

$$J = \int_{t_0}^{t_1} f_0(\underset{\sim}{x}, \underset{\sim}{u})dt \qquad \text{(cost of transfer)}.$$

State Equations

$$\dot{x}_i = f_i(\underset{\sim}{x}, \underset{\sim}{u}), \quad i = 1, \ldots, 4.$$

Cost Functional

$$J = \int_{t_0}^{t_1} f_0(\underset{\sim}{x}, \underset{\sim}{u})dt$$

One **General Class** of problem: Subject to the DE constraint

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u})$$

control $x(t)$, an n-dimensional vector, from $\underset{\sim 0}{x}$ at $t = t_0$ to $\underset{\sim 1}{x}$ at $t = t_1$ so that the cost functional

$$J = \int_{t_0}^{t_1} f_0(\underset{\sim}{x}, \underset{\sim}{u}) dt$$

is minimized.

$$\underset{\sim 0}{x}, \underset{\sim 1}{x}, t_0 \quad \text{FIXED}; \quad t_1 \quad \text{UNSPECIFIED.}$$

An admissible control $\underset{\sim}{u} = \underset{\sim}{u}^*(t)$ that transfers the system from $\underset{\sim 0}{x}$ to $\underset{\sim 1}{x}$ and minimizes J is called an OPTIMAL CONTROL. We assume that such a control exists. A first necessary condition analogous to the Euler-Lagrange equation, is needed. The idea is that any control failing this condition could not be optimal. Any control satisfying it *might be* optimal and needs to be looked at more closely.

Given

- State eqns $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u});$
- Initial state $\underset{\sim}{x}(t_0) = \underset{\sim 0}{x};$
- Cost functional $J = \int_{t_0}^{t_1} f_0(\underset{\sim}{x}, \underset{\sim}{u}) dt;$
- Set of admissible controls $\underset{\sim}{u}(t) \in U$ (piecewise conts.)

Find $\underset{\sim}{u}^*(t)$ and path $\underset{\sim}{x}^*(t)$, transferring system from $\underset{\sim 0}{x}$ to $\underset{\sim 1}{x}$ such that J is minimized.

Pontryagin Maximum Principle

illustrated for a $2-D$ state space. Consider

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, u). \end{aligned}$$

Want to go from (x_1^0, x_2^0) at $t = t_0$ to (x_1^1, x_2^1) at time t_1, using **(#) piecewise smooth, (##) bounded** admissible control functions $u(t)$ so that

$$J = \int_{t_0}^{t_1} f_0(x_1, x_2, u) dt \quad \text{minimized.}$$

Hamiltonian

$$H = \psi_0 f_0(x_1, x_2, u) + \psi_1 f_1(x_1, x_2, u) + \psi_2 f_2(x_1, x_2, u)$$

where $\psi_i = \psi_i(t)$, $i = 0, 1, 2$, satisfy

$$\dot{\psi}_i = -\partial H / \partial x_i\,, \quad i = 0, 1, 2$$

(x_0 defined below)

Theorem. Let $u^*(t)$ be an admissible control with corresponding path $\underset{\sim}{x}^*(t) = \binom{x_1^*}{x_2^*}$ that takes the system from $(t_0, \underset{\sim}{x}^0)$ to $(t_1, \underset{\sim}{x}^1)$ at some (unspecified) time t_1. Then in order that u^* and $\underset{\sim}{x}^*$ be optimal, it is necessary that $\exists$ a non zero vector $\underset{\sim}{\psi} = (\psi_0, \psi_1, \psi_2)^T$ such that

(PMP) $\qquad \underset{\sim i}{\psi} = -\partial H / \partial x_i\,, \quad i = 0, 1, 2 \quad$ Costate eqns

and a scalar function

$$H(\underset{\sim}{\psi}, \underset{\sim}{x}, u) = \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2$$

such that

(i) $\forall t \in [t_0, t_1]$, H attains its maximum as a function of u at $u = u^*(t)$;

(ii) $H(\underset{\sim}{\psi}^*, \underset{\sim}{x}^*, \underset{\sim}{u}^*) = 0$ and $\psi_0 \le 0$ at $t = t_1$, where $\underset{\sim}{\psi}^*(t)$ is the solution of (PMP) for $u = u^*(t)$.

Furthermore, $H(\underset{\sim}{\psi}^*(t), \underset{\sim}{x}^*(t),\ u^*(t)) = \text{const.}$ and $\psi_0(t) = \text{const.}$, so that $H = 0$ and $\psi_0 \le 0$ at each point on an optimal trajectory. This theorem called Pontryagin Maximum Princ.

DEFINITION of x_0: Let x_0 be the solution of the DE

$\dot{x}_0 = f_0(x_1, x_2, u)$, $x_0(t_0) = 0$.

That is, $x_0(t)$ measures the cost incurred up to t.

Example 1. Let $\dot{x}_1 = -x_1 + u$. Control the system from $x_1 = a$ at $t = 0$ to $x_1 = b$ at t_1, so that $J = \frac{1}{2}\int_0^{t_1} u^2 dt$ is minimized. We will consider the special cases $a = 1,\ b = 2$ and $b = 2,\ a = 1$.

Solution. Either write down the previous equations omitting x_2, f_2, and ψ_2 or let x_2 be a constant by:

$$\begin{aligned}
f_0(x_1, x_2, u) &= \frac{1}{2}u^2 \\
f_1(x_1, x_2, u) &= -x_1 + u \\
f_2(x_1, x_2, u) &= 0.
\end{aligned}$$

$$\begin{aligned}
H &= \psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2 = \psi_0\, u^2/2 + \psi_1(-x_1 + u) \\
\dot{\psi}_0 &= -\partial H/\partial x_0 = 0 \Rightarrow \psi_0 = \text{ const.} \\
\dot{\psi}_1 &= -\partial H/\partial x_1 = \psi_1.
\end{aligned}$$

Always $\psi_0 =$ const. because H is not a function of x_0

So choose $\psi_0 = -1$

(as long as $\psi_0 < 0$ we can divide everything by ψ_0).

So $\psi = -1$, $\psi_1 = Ae^t$.

Now maximize H as a function of u:

$$\partial H/\partial u = \psi_0 u + \psi_1 = -u + \psi_1,\ \frac{\partial^2 H}{\partial u^2} = -1.$$

So H is minimized in u by

$$u = \psi_1,$$

That is $u = Ae^t$. The corresponding soln for x is

$$\begin{aligned} \dot{x}_1 &= -x_1 + Ae^t \\ \Rightarrow \quad x_1(t) &= Be^{-t} + \frac{1}{2}Ae^t. \end{aligned}$$

END CONDITIONS:

$$x_1(0) = a = B + A/2$$

$$x_1(t_1) = b = Be^{-t_1} + Ae^{t_1}/2$$

$$\Rightarrow \quad B = \tfrac{ae^{t_1}-b}{e^{t_1}-e^{-t_1}}, \quad \tfrac{A}{2} = \tfrac{b-ae^{-t_1}}{e^{t_1}-e^{-t_1}}.$$

Now t_1 free so that $H(t_1) = 0$ and H does not depend explicitly on t so that $H \equiv K$, a constant. Thus $H(t) = 0$ for all t. Now $H = u^2/2 + \psi_1(-x_1 + u)$ so $H(0) = A^2/2 + A(-[A/2 + B] + A) = -AB$, since $u^*(0) = \psi_1(0) = A$ and $x_1(0) = A/2 + B$. $a = 2$, $b = 1$: Then $AB = 0 \quad \Rightarrow \quad A = 0$ or $B = 0$. If $B = 0$, $e^{t_1} = 1/2$, impossible for $t_1 > 0$. If $A = 0$, $e^{t_1} = 2$. Then the optimal control is $u = 0$, optimal path $x_1 = 2e^{-t}$, total cost $J = 0$. $a = 1, b = 2$: Then $AB = 0 \quad \Rightarrow \quad A = 0$ or $B = 0$. If $B = 0$,

$e^{t_1} = 2$ and $t_1 = \ln 2$. Then $A = 2$ so the optimal control is $u = 2e^t$, optimal path is $x_1 = e^t/2$ total cost $J = \int_0^{\ln 2} 2e^{2t} dt = 3$. If $A = 0$, $e^{t_1} = 1/2$ impossible for $t_1 > 0$. So it costs less for the system to go from $x_0 = 2$ to $x_1 = 1$, by exponential decay, than from $x_0 = 1$ to $x_1 = 2$ by forcing. However, if $|u| \leq 1$, we could not get from $x_0 = 1$ to $x_1 = 2$ at all. In fact, keeping $u = +1$ just holds the system at the level $x_0 = 1$.

Example 2. $\dot{x}_1 = x_2$, $\dot{x}_2 = u$; control from given initial point (a, b) at $t_0 = 0$ to $\underset{\sim}{0}$ in as short a time as possible.

Solution. "Time-optimal control to the origin". Note the system is just $\ddot{x} = u$, $x = a$, $\dot{x} = b$ at $t = t_0 = 0$.

A physical model

is a propelled truck

$m\ddot{x} = F, \quad |F| \le K$

$\ddot{x} = u(t)$, $|u| \le K/m$.

Cost is t_1 and has to be in the form

$$J = \int_0^{t_1} f_0(x_1, x_2, u)dt$$

to use the PMP. Choose $f_0 = 1$

$$\begin{aligned} f_0(x_1, x_2, u) &= 1 \\ f_1(x_1, x_2, u) &= x_2 \\ f_2(x_1, x_2, u) &= u \end{aligned}$$

$$H = -1 + \psi_1 x_2 + \psi_2 u$$

$$\dot{\psi}_1 = -\partial H/\partial x_1 = 0\,; \quad \dot{\psi}_2 = -\partial H/\partial x_2 = -\psi_1$$

$$\Rightarrow \quad \psi_1 = A, \quad \psi_2 = B - At.$$

Now maximize H as a function of u, where u is taken as bounded, $|u| \le K/m$. The maximum value of H if

$\psi_2 > 0$ is for $u = K/m$; if $\psi_2 < 0$, max value occurs when $u = -K/m$.

$$u(t) = \frac{K}{m} \text{ sqn } \psi_2(t)$$

That is, the control *switches* from K/m to $-K/m$ when ψ_2 goes from positive to negative; and switches from $-K/m$ to K/m when ψ_2 changes from negative to positive.

But $\psi_2 = B - At$ can change sign once
$\Rightarrow$ only one switch.

Corresponding solution for x_1 and x_2 is as follows: let $u^*(t) = \frac{K}{m} \operatorname{sgn} \psi_2(t)$. This is constant between switches,

$+K/m$ or $-K/m$:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u^*,$$

$$u^* = \pm K/m$$

$$dx_2/dx_1 = u^*/x_2$$

$$\int x_2 dx_2 = \int u^* dx_1 + \text{ const}$$

$$\text{i.e.} \quad x_2^2 = 2u^* x_1 + \text{ const.}$$

To each of the allowed values of u^* there corresponds a family of parabolas.

$u^* = K/m$ $\qquad$ $u^* = -K/m$